Integration

Integration is the reverse process of differentiation. When we differentiate we start with an expression and proceed to find its derivative. When we integrate, we start with the derivative and then find the expression from which it has been derived.

Indefinite Integral

If $\frac{d}{dx}F(x) = f(x)$, then:

$\int f(x)\,dx = F(x) + c$, where c is a constant of integration.

Theorems

$(1)\ \int [f_1(x) \pm f_2(x) \pm \cdots \pm f_n(x)]dx = \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \cdots \pm \int f_n(x)\,dx$

$(2)\ \int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a constant

$(3)\ \int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

$(4)\ \int [f(x)]^n f'(x)\,dx = \frac{f^{n+1}(x)}{n+1} + c,\ n \neq -1$

Ex1: Evaluate:

$(1)\ \int x^5\,dx = \frac{x^6}{6} + c$

$(2)\ \int \left(x^3 + 2x + \frac{1}{x^2} + 4\right)dx = \int \left(x^3 + 2x + x^{-2} + 4\right)dx = \frac{x^4}{4} + 2\frac{x^2}{2} + \frac{x^{-1}}{-1} + 4x + c$

$(3)\ \int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int 2x\left(1+x^2\right)^{1/2}dx = \frac{1}{2}\frac{\left(1+x^2\right)^{3/2}}{3/2} + c = \frac{\left(1+x^2\right)^{3/2}}{3} + c$

$(4)\ \int (4x-3)^6\,dx = \frac{1}{4}\int 4(4x-3)^6\,dx = \frac{1}{4}\frac{(4x-3)^7}{7} + c = \frac{(4x-3)^7}{28} + c$

Integration of Exponential and Logarithmic Functions

$(1)\ \int \frac{1}{x}\,dx = \ln|x| + c$ $\qquad$ $(2)\ \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$(3)\ \int e^x\,dx = e^x + c$ $\qquad$ $(4)\ \int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$

Ex2: Evaluate:

$$(1)\ \int e^{(5x-6)}\,dx = \frac{1}{5}\int 5e^{5x-6}\,dx = \frac{1}{5}e^{5x-6} + c = \frac{e^{5x-6}}{5} + c$$

$$(2)\ \int e^{x^2+2x}(x+1)dx = \frac{1}{2}\int 2(x+1)e^{x^2+2x}\,dx = \frac{e^{x^2+2x}}{2} + c$$

$$(3)\ \int 2^{3x+1}\,dx = \int e^{\ln 2^{3x+1}}\,dx = \int e^{\ln 2(3x+1)}\,dx = \frac{1}{3\ln 2}\int 3\ln 2\, e^{\ln 2(3x+1)}\,dx$$

$$= \frac{1}{3\ln 2}e^{\ln 2(3x+1)} + c = \frac{2^{3x+1}}{3\ln 2} + c$$

$$(4)\ \int \frac{1}{1-2x}\,dx = \frac{1}{-2}\int \frac{-2}{1-2x}\,dx = \frac{-1}{2}\ln|1-2x| + c = \frac{-\ln|1-2x|}{2} + c$$

$$(5)\ \int \frac{x^3}{1+x^4}\,dx = \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx = \frac{\ln(1+x^4)}{4} + c$$

$$(6)\ \int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx = \ln|\ln x| + c$$

Integration of Trigonometric and Inverse Trigonometric Functions

$(1)\ \int \sin x\,dx = -\cos x + c$ $\quad$ $(2)\ \int \sin[f(x)]f'(x)\,dx = -\cos[f(x)] + c$

$(3)\ \int \cos x\,dx = \sin x + c$ $\quad$ $(4)\ \int \cos[f(x)]f'(x)\,dx = \sin[f(x)] + c$

$(5)\ \int \sec^2 x\,dx = \tan x + c$ $\quad$ $(6)\ \int \sec^2[f(x)]f'(x)\,dx = \tan[f(x)] + c$

$(7)\ \int \csc^2 x\,dx = -\cot x + c$ $\quad$ $(8)\ \int \csc^2[f(x)]\,f'(x)\,dx = -\cot[f(x)] + c$

$(9)\ \int \sec x \tan x dx = \sec x + c$ $\quad$ $(10)\ \int \sec[f(x)]\tan[f(x)]f'(x)\,dx = \sec[f(x)] + c$

$(11)\ \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$ $\quad$ $(12)\ \int \frac{f'(x)}{\sqrt{1-f^2(x)}}\,dx = \sin^{-1}[f(x)] + c$

$(11)\ \int \frac{1}{\sqrt{x^2-1}}\,dx = \cos^{-1} x + c$ $\quad$ $(12)\ \int \frac{f'(x)}{\sqrt{f^2(x)-1}}\,dx = \cos^{-1}[f(x)] + c$

$(13)\ \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$ $\quad$ $(14)\ \int \frac{f'(x)}{1+f^2(x)}\,dx = \tan^{-1}[f(x)] + c$

$$(15)\ \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c \qquad (16)\ \int \frac{f'(x)}{f(x)\sqrt{f^2(x)-1}}\,dx = \sec^{-1}[f(x)] + c$$

$$(17)\ \int \frac{1}{x\sqrt{1-x^2}}\,dx = \csc^{-1} x + c \qquad (18)\ \int \frac{f'(x)}{f(x)\sqrt{1-f^2(x)}}\,dx = \csc^{-1}[f(x)] + c$$

<u>Ex3:</u> Determine the following integrals:

$$(1)\ \int 3\sin(2x+3)\,dx = \frac{3}{2}\int 2\sin(2x+3)\,dx = -\frac{3}{2}\cos(2x+3) + c$$

$$(2)\ \int 6\sec^2(5-3x)\,dx = \frac{6}{-3}\int -3\sec^2(5-3x)\,dx = -2\tan(5-3x) + c$$

$$(3)\ \int \frac{2}{4+x^2}\,dx = \frac{2}{4}\int \frac{1}{1+(x/2)^2}\,dx = \int \frac{1/2}{1+(x/2)^2}\,dx = \tan^{-1}(x/2) + c$$

$$(4)\ \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int \sin^{-1} x \frac{1}{\sqrt{1-x^2}}\,dx = \frac{(\sin^{-1} x)^2}{2} + c$$

$$(5)\ \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx = \sin^{-1}(e^x) + c$$

$$(6)\ \int \frac{1}{e^x + e^{-x} + 2}\,dx = \int \frac{e^x}{e^{2x} + 1 + 2e^x}\,dx = \int \frac{e^x}{(e^x+1)^2}\,dx$$

$$= \int (e^x+1)^{-2} e^x\,dx = \frac{(e^x+1)^{-1}}{-1} + c = \frac{-1}{e^x+1} + c$$

$$(7)\ \int \frac{\cos x}{5-\cos^2 x}\,dx = \int \frac{\cos x}{5-1+\sin^2 x}\,dx = \int \frac{\cos x}{4+\sin^2 x}\,dx$$

$$= \frac{1}{4}\int \frac{\cos x}{1+\left(\frac{\sin x}{2}\right)^2}\,dx = \frac{1}{2}\int \frac{(\cos x)/2}{1+\left(\frac{\sin x}{2}\right)^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + c$$

$$(8)\ \int \frac{1}{x\sqrt{4x^2-3}}\,dx = \int \frac{1}{x\sqrt{3\left(\frac{4}{3}x^2-1\right)}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{x\sqrt{\left(\frac{2x}{\sqrt{3}}\right)^2-1}}\,dx$$

$$=\frac{1}{\sqrt{3}}\int\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x}{\sqrt{3}}\right)\sqrt{\left(\frac{2x}{\sqrt{3}}\right)^{2}-1}}dx=\frac{1}{\sqrt{3}}\sec^{-1}\left(\frac{2x}{\sqrt{3}}\right)+c$$

$$(9)\ \int\frac{1}{\sqrt{x}\,(1+x)}dx=\int\frac{1/\sqrt{x}}{1+\left(\sqrt{x}\right)^{2}}dx=2\int\frac{1/\left(2\sqrt{x}\right)}{1+\left(\sqrt{x}\right)^{2}}dx=2\tan^{-1}\sqrt{x}+c$$

$$(10)\ \int\frac{1}{x\sqrt{3-(\ln x)^{2}}}dx=\int\frac{\frac{1}{\sqrt{3}\,x}}{\sqrt{1-\left(\ln x/\sqrt{3}\right)^{2}}}dx=\sin^{-1}\left(\frac{\ln x}{\sqrt{3}}\right)+c$$

Ex4: Integrate:

$$(1)\ \int\tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\int\frac{-\sin x}{\cos x}dx=-\ln|\cos x|+c=\ln\left|\frac{1}{\cos x}\right|+c=\ln|\sec x|+c$$

$$(2)\ \int\tan x\sec^{2}x\,dx=\frac{\tan^{2}x}{2}+c$$

$$(3)\ \int\sin x\cos x\,dx=\frac{\sin^{2}x}{2}+c$$

Or $\int\sin x\cos x\,dx=-\int\cos x\,(-\sin x)dx=\frac{-\cos^{2}x}{2}+c$

Or $\int\sin x\cos x\,dx=\int 0.5\sin 2x\,dx=\frac{-0.5\cos 2x}{2}+c=\frac{-\cos 2x}{4}+c$

$$(4)\ \int\frac{1}{1+\sin x}dx=\int\frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}dx=\int\frac{1-\sin x}{1-\sin^{2}x}dx$$

$$=\int\frac{1-\sin x}{\cos^{2}x}dx=\int\frac{1}{\cos^{2}x}dx-\int\frac{\sin x}{\cos^{2}x}dx=\int\sec^{2}x\,dx+\int\cos^{-2}x\,(-\sin x)dx$$

$$=\tan x+\frac{(\cos x)^{-1}}{-1}+c=\tan x-\frac{1}{\cos x}+c=\tan x-\sec x+c$$

$$(5)\ \int\frac{\cos x}{1-\cos x}dx=\int\frac{\cos x}{1-\cos x}\times\frac{1+\cos x}{1+\cos x}dx=\int\frac{\cos x+\cos^{2}x}{1-\cos^{2}x}dx$$

$$=\int\frac{\cos x+\cos^{2}x}{\sin^{2}x}dx=\int\frac{\cos x+1-\sin^{2}x}{\sin^{2}x}dx$$

$$= \int (\sin x)^{-2} \cos x \, dx + \int \csc^2 x \, dx - \int dx = \frac{(\sin x)^{-1}}{-1} - \cot x - x + c$$

$$= -\csc x - \cot x + c$$

$$(6) \int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

$$(7) \int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1-\sin^2 x)\cos x \, dx$$

$$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx = \sin x - \frac{\sin^3 x}{3} + c$$

$$(8) \int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx = \int (\sec^2 x - 1)\tan x \, dx$$

$$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx = \frac{\tan^2 x}{2} - \ln|\sec x| + c$$

$$(9) \int \sqrt{1+\sin 2x} \, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} \, dx$$

$$= \int \sqrt{(\sin x + \cos x)^2} \, dx = \int (\sin x + \cos x) dx = \int \sin x \, dx + \int \cos x \, dx$$

$$= -\cos x + \sin x + c$$

$$(10) \int \frac{\sin 2x}{\sin x} dx = \int \frac{2\sin x \cos x}{\sin x} dx = \int 2\cos x \, dx = 2\sin x + c$$

$$(11) \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = 2\int \sin\sqrt{x} \times \frac{1}{2\sqrt{x}} dx = -2\cos\sqrt{x} + c$$

$$(12) \int \frac{\sin x}{\csc x + \cot x} dx = \int \frac{\sin x}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} dx = \int \frac{\sin^2 x}{1+\cos x} dx = \int \frac{1-\cos^2 x}{1+\cos x} dx$$

$$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx = \int (1-\cos x) dx = x - \sin x + c$$

$$(13) \int \sin 5x \sin 3x \, dx$$

$$\because \cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\because \cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

$$\therefore \cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2) = 2\sin\theta_1 \sin\theta_2$$

$$\therefore \sin\theta_1 \sin\theta_2 = \frac{\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)}{2}$$

$$\therefore \sin 5x \sin 3x = \frac{\cos(5x-3x)-\cos(5x+3x)}{2} = \frac{\cos 2x - \cos 8x}{2}$$

$$\therefore \int \sin 5x \sin 3x\, dx = \frac{1}{2}\int \cos 2x\, dx - \frac{1}{2}\int \cos 8x\, dx = \frac{1}{2}\frac{\sin 2x}{2} - \frac{1}{2}\frac{\sin 8x}{8} + c$$

$$= \frac{\sin 2x}{4} - \frac{\sin 8x}{16} + c$$

Definite Integral

If $\int f(x)\,dx = F(x) + c$, then:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Ex5: Evaluate the following integrals

$$(1)\ \int_0^1 (2x+1)^3\,dx = \left[\frac{1}{2}\frac{(2x+1)^4}{4}\right]_0^1 = \left[\frac{(2(1)+1)^4}{8}\right] - \left[\frac{(2(0)+1)^4}{8}\right] = \frac{81}{8} - \frac{1}{8} = \frac{80}{8} = 10$$

$$(2)\ \int_0^{\pi/2} \sin^3 x \cos^3 x\,dx = \int_0^{\pi/2} \sin^3 x \cos^2 x \cos x\,dx = \int_0^{\pi/2} \sin^3 x (1-\sin^2 x)\cos x\,dx$$

$$\int_0^{\pi/2} (\sin^3 x - \sin^5 x)\cos x\,dx = \int_0^{\pi/2} \sin^3 x \cos x\,dx - \int_0^{\pi/2} \sin^5 x \cos x\,dx = \left[\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6}\right]_0^{\pi/2}$$

$$= \left(\frac{1}{4} - \frac{1}{6}\right) - \left(\frac{0}{4} - \frac{0}{6}\right) = \frac{3-2}{12} = \frac{1}{12}$$

$$(3)\ \int_0^{\pi/4} \sec x\,dx = \int_0^{\pi/4} \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int_0^{\pi/4} \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0| = \ln(\sqrt{2}+1)$$

$$(4)\ \int_0^{\pi/4} \cos^4 2x\,dx = \int_0^{\pi/4} \cos^2 2x \times \cos^2 2x\,dx = \int_0^{\pi/4} \left(\frac{1+\cos 4x}{2}\right)\left(\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\int_0^{\pi/4} (1+\cos 4x)^2\,dx = \frac{1}{4}\int_0^{\pi/4} (1 + 2\cos 4x + \cos^2 4x)\,dx$$

$$=\frac{1}{4}\int_0^{\pi/4}(1+2\cos 4x+0.5+0.5\cos 8x)\,dx=\frac{1}{4}\int_0^{\pi/4}(1.5+2\cos 4x+0.5\cos 8x)\,dx$$

$$=\frac{1}{4}\left(1.5x+2\frac{\sin 4x}{4}+0.5\frac{\sin 8x}{8}\right)_0^{\pi/4}=\frac{1}{4}\left(1.5\frac{\pi}{4}+\frac{\sin\pi}{2}+\frac{\sin 2\pi}{16}\right)=\frac{3\pi}{32}$$

Properties of Definite Integral

$$(1)\ \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx \qquad (2)\ \int_a^a f(x)\,dx=0$$

$$(3)\ \int_a^b f(x)\,dx=\int_a^d f(x)\,dx+\int_d^b f(x)\,dx$$

$$(4)\ \frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

$$(5)\ \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt=f[v(x)]v'(x)-f[u(x)]u'(x)$$

Ex6: Find $\frac{dy}{dx}$, if:

$$(1)\ y=\int_{x^2}^{2}\frac{dt}{1+t}=-\int_2^{x^2}\frac{1}{1+t}dt \ \Rightarrow\ \frac{dy}{dx}=\frac{-1}{1+x^2}\frac{d}{dx}x^2=\frac{-2x}{1+x^2}$$

$$(2)\ y=\int_{\cos x}^{\sin x}\sqrt{1-t^2}\,dt \ \Rightarrow\ \frac{dy}{dx}=\sqrt{1-\sin^2 x}\cos x-\sqrt{1-\cos^2 x}\,(-\sin x)$$

$$\therefore\ \frac{dy}{dx}=\sqrt{\cos^2 x}\cos x+\sqrt{\sin^2 x}\sin x=\cos^2 x+\sin^2 x=1$$

Ex7: Variables x and y are related by the equation

$$x=\int_0^y\frac{1}{\sqrt{1+4t^2}}dt\cdot$$

Show that $\frac{d^2y}{dx^2}$ is proportional to y and find the constant of proportionality.

Solution

$$\because\ x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt \Rightarrow \frac{d}{dx}x = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt$$

$$\therefore 1 = \frac{1}{\sqrt{1+4y^2}}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+4y^2}}8y\frac{dy}{dx} = \frac{4y}{\sqrt{1+4y^2}}\frac{dy}{dx} = \frac{4y}{\sqrt{1+4y^2}}\times\sqrt{1+4y^2} = 4y$$

$$\therefore \frac{d^2y}{dx^2} = 4y \Rightarrow \frac{d^2y}{dx^2} = ky \text{ and } k = 4$$

Home works (2)

Q1: Determine the followings:

(1) $\int \cot x \, dx$

(2) $\int \csc x \, dx$

(3) $\int \cos^5 x \, dx$

(4) $\int \cot^3 x \, dx$

(5) $\int \cos 5x \; \sin x \, dx$

(6) $\int \cos 3x \; \cos x \, dx$

(7) $\int \frac{\cos x}{\sec x + \tan x} dx$

(8) $\int \frac{\sin 2x}{1 + \cos^2 x} dx$

(9) $\int \tan^2 x \sec^2 x \, dx$

(10) $\int \sin^2 x \cos^2 x \, dx$

(11) $\int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} d\theta$

(12) $\int \frac{y}{2 + y^4} dy$

(13) $\int \frac{e^{2x}}{e^{2x} + e^{-2x}} dx$

(14) $\int \frac{1}{e^t + e^{-t}} dt$

Q2: Evaluate the following integrals:

(1) $\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$

(2) $\int_0^2 \frac{2}{8 + 3x^2} dx$

(3) $\int_0^{\pi/2} \sin^5 x \cos^3 x \, dx$

(4) $\int_{\pi/8}^{\pi/6} \ln(\sin 2x) \cot 2x \, dx$

(5) $\int_0^3 \frac{3}{\sqrt{9 - x^2}} dx$

(6) $\int_0^{\pi/2} \frac{\cos 2x}{\cos x + \sin x} dx$

Q3: Find $\frac{dy}{dx}$, if:

(1) $y = \int_x^{2x} \frac{dt}{t - 2}$

(2) $y = \int_{\cos x}^{\sin x} \frac{dt}{1 - t^2}$

Q4: Given:

$$y = \int_0^x f(t)(x - t) dt \cdot show\ that\ \frac{d^2 y}{dx^2} = f(x) \cdot$$

Q5: Variables x and y are related by the equation

$$8x = \int_0^y \frac{1}{\sqrt{1 + t}} dt \cdot \text{ Show that } \frac{d^2 y}{dx^2} = 32 \cdot$$