Methods of Integration

1- Integration by Parts

We often need to integrate a product where either function is not the derivative of the other. For example, in the case of

$$\int x \ln x \, dx$$

$\ln x$ is not the derivative of x

x is not the derivative of $\ln x$

so in the situation like this, we have to find some other method of dealing with the integral. Let us establish the rule for such cases.

If u and v are functions of x, then we know that

$$\frac{d}{dx}(u\,v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Now integrating both sides with respect to x. On the left, we get back to the function from which we started:

$$u\,v = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

and rearranging the terms, we have:

$$\int u\frac{dv}{dx}dx = u\,v - \int v\frac{du}{dx}dx$$

$$\therefore \int u\,dv = u\,v - \int v\,du$$

Ex1: Evaluate the following integrals:

$(1)\ \int x \ln x \, dx$

$$Let\ u = \ln x \Rightarrow \ du = \frac{1}{x}dx$$

$$and\ \ dv = x\,dx \Rightarrow v = \frac{x^2}{2}$$

$$\because \int u\,dv = u\,v - \int v\,du$$

$$\therefore \int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$(2) \int x \sin x \, dx$

$Let\ u = x \Rightarrow \quad du = dx$

$and \quad dv = \sin x\, dx \Rightarrow v = -\cos x$

$$\therefore \int x \sin x \, dx = -x\cos x - \int -\cos x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + c$$

$(3) \int x^2 e^x \, dx$

$Let\ u = x^2 \Rightarrow \quad du = 2x\, dx$

$and \quad dv = e^x\, dx \Rightarrow v = e^x$

$$\therefore \int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - I + c$$

$I = \int 2x e^x \, dx$

$Let\ u = 2x \Rightarrow \quad du = 2\, dx$

$and \quad dv = e^x\, dx \Rightarrow v = e^x$

$$\therefore \int 2x e^x \, dx = 2x e^x - \int 2 e^x \, dx = 2x e^x - 2e^x$$

$$\therefore \int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + c$$

$(4) \int e^x \cos 2x \, dx$

$Let\ u = e^x \quad \Rightarrow du = e^x\, dx$

$and \quad dv = \cos 2x\, dx \quad \Rightarrow \quad v = \frac{\sin 2x}{2}$

$$\therefore \int e^x \cos 2x \, dx = e^x \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} e^x \, dx = \frac{e^x \sin 2x}{2} - I$$

$I = \int \frac{\sin 2x}{2} e^x \, dx$

$Let\ u = e^x \Rightarrow du = e^x\, dx$

$and \quad dv = \frac{\sin 2x}{2} dx \quad \Rightarrow v = \frac{-\cos 2x}{4}$

$$\therefore \int \frac{\sin 2x}{2} e^x\,dx = \frac{-e^x \cos 2x}{4} - \int \frac{-\cos 2x}{4} e^x\,dx = \frac{-e^x \cos 2x}{4} + \int \frac{\cos 2x}{4} e^x\,dx$$

$$\therefore \int e^x \cos 2x\,dx = \frac{e^x \sin 2x}{2} + \frac{e^x \cos 2x}{4} - \frac{1}{4}\int e^x \cos 2x\,dx$$

$$\therefore \left(1+\frac{1}{4}\right)\int e^x \cos 2x\,dx = \frac{e^x \sin 2x}{2} + \frac{e^x \cos 2x}{4}$$

$$\therefore \frac{5}{4}\int e^x \cos 2x\,dx = \frac{e^x \sin 2x}{2} + \frac{e^x \cos 2x}{4} \quad \times \frac{4}{5}$$

$$\therefore \int e^x \cos 2x\,dx = \frac{2e^x \sin 2x}{5} + \frac{e^x \cos 2x}{5} + c$$

$(5)\ \int \tan^{-1} x\,dx$

$Let\ u = \tan^{-1} x \Rightarrow \quad du = \frac{1}{1+x^2}dx$

$and \quad dv = dx \Rightarrow v = x$

$$\therefore \int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + c$$

$(6)\ \int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$

$Let\ u = \sec x \Rightarrow \quad du = \sec x \tan x\,dx$

$and \quad dv = \sec^2 x\,dx \Rightarrow v = \tan x$

$$\therefore \int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$\because 1+\tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$$

$$\therefore \int \sec^3 x\,dx = \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right)dx$$

$$\therefore \int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore 2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

$$\therefore \int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx = \frac{\sec x \tan x}{2} + \frac{\ln\left|\sec x + \tan x\right|}{2} + c$$

<u>2- Integration by Partial Fractions</u>

<u>Ex2:</u> Evaluate the following integrals:

(1) $\int \frac{x}{(x-1)(x-2)}dx$

$$\therefore \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$\therefore A = -1$ and $B = 2$

$$\therefore \int \frac{x}{(x-1)(x-2)}dx = \int \frac{-1}{x-1}dx + \int \frac{2}{x-2}dx = -\ln|x-1| + 2\ln|x-2| + c$$

(2) $\int \frac{4x^2}{(x+1)(x-1)^2}dx$

$$\therefore \frac{4x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$\therefore A = 1,\ B = 3,$ and $C = 2$

$$\therefore \int \frac{4x^2}{(x+1)(x-1)^2}dx = \int \frac{1}{x+1}dx + \int \frac{3}{x-1}dx + \int \frac{2}{(x-1)^2}dx$$

$$= \ln|x+1| + 3\ln|x-1| + 2\frac{(x-1)^{-1}}{-1} + c = \ln|x+1| + 3\ln|x-1| - \frac{2}{x-1} + c$$

(3) $\int \frac{x^2+1}{(x+2)^3}dx$

$$\therefore \frac{x^2+1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

$\therefore A = 1,\ B = -4,$ and $C = 5$

$$\therefore \int \frac{x^2+1}{(x+2)^3}dx = \int \frac{1}{x+2}dx + \int \frac{-4}{(x+2)^2}dx + \int \frac{5}{(x+2)^3}dx$$

$$= \ln|x+2| - 4\frac{(x+2)^{-1}}{-1} + 5\frac{(x+2)^{-2}}{-2} + c = \ln|x+2| + \frac{4}{x+2} - \frac{5}{2(x+2)^2} + c$$

(4) $\int \frac{x^2+3x+2}{x(x^2+1)}dx$

$$\therefore \frac{x^2+3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} \quad \times\ x(x^2+1)$$

$$\therefore\ x^2+3x+2=A\left(x^2+1\right)+Bx^2+Cx=Ax^2+A+Bx^2+Cx$$

$\therefore\ A=2,\ B=-2,$ and $C=3$

$$\therefore\ \int\frac{x^2+3x+2}{x\left(x^2+1\right)}dx=\int\frac{2}{x}dx+\int\frac{-2x+3}{x^2+1}dx=\int\frac{2}{x}dx-\int\frac{2x}{x^2+1}dx+\int\frac{3}{x^2+1}dx$$

$$=2\ln|x|-\ln\left(x^2+1\right)+3\tan^{-1}x+c$$

(5) $\int\frac{2x^2+x+5}{x(2x+1)}dx$

$$\therefore\ \frac{2x^2+x+5}{x(2x+1)}=A+\frac{B}{x}+\frac{C}{(2x+1)}\quad\times\ x(2x+1)$$

$$\therefore\ 2x^2+x+5=A\left(2x^2+x\right)+B(2x+1)+Cx=2Ax^2+Ax+2Bx+B+Cx$$

$\therefore\ A=1,\ B=5,$ and $C=-10$

$$\therefore\ \int\frac{2x^2+x+5}{x(2x+1)}dx=\int dx+\int\frac{5}{x}dx+\int\frac{-10}{2x+1}dx=\int dx+\int\frac{5}{x}dx-5\int\frac{2}{2x+1}dx$$

$$=x+5\ln|x|-5\ln(2x+1)+c$$

<u>3- Integration by Substitution</u>

Make the substitutions:

(1) $x=a\sin\theta$ or $x=a\tanh\theta$

in integrals involving $\sqrt{a^2-x^2}$ or $\left(a^2-x^2\right)$.

(2) $x=a\tan\theta$ or $x=a\sinh\theta$

in integrals involving $\sqrt{a^2+x^2}$ or $\left(a^2+x^2\right)$.

(3) $x=a\sec\theta$ or $x=a\cosh\theta$

in integrals involving $\sqrt{x^2-a^2}$ or $\left(x^2-a^2\right)$.

<u>Ex3:</u> Find:

(1) $\int\frac{dx}{\sqrt{4-x^2}}=\int\frac{(1/2)}{\sqrt{1-(x/2)^2}}dx=\sin^{-1}(x/2)+c$

This integral can be solved by substitution as follows:

Let $x=2\sin\theta\Rightarrow dx=2\cos\theta\,d\theta$ and $4-x^2=4-4\sin^2\theta=4\cos^2\theta$

$$\therefore \sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\therefore \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta = \theta + c$$

$$\because\ x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}(x/2)$$

$$\therefore \int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}(x/2) + c$$

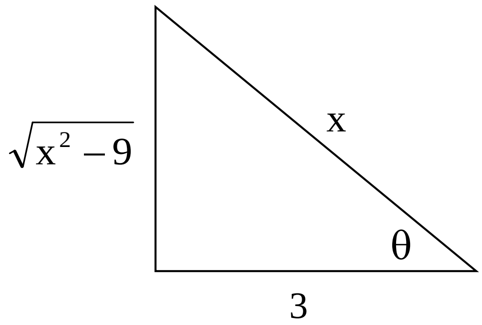
(2) $\int \frac{dx}{\sqrt{x^2-9}}$

Let $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9}$

$$\therefore \sqrt{x^2-9} = \sqrt{9\tan^2\theta} = 3\tan\theta$$

$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$\because\ x = 3\sec\theta \Rightarrow \sec\theta = \frac{x}{3} \text{ and } \tan\theta = \frac{\sqrt{x^2-9}}{3}$$



$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + c$$

Or $x = 3\cosh\theta \Rightarrow dx = 3\sinh\theta\, d\theta$ and $\sqrt{x^2-9} = \sqrt{9\cosh^2\theta - 9}$

$$\therefore \sqrt{x^2-9} = \sqrt{9\sinh^2\theta} = 3\sinh\theta$$

$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sinh\theta\, d\theta}{3\sinh\theta} = \int d\theta = \theta + c = \cosh^{-1}\frac{x}{3} + c$$

$$\because \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \Rightarrow \cosh^{-1}\frac{x}{3} = \ln\left(\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right)$$

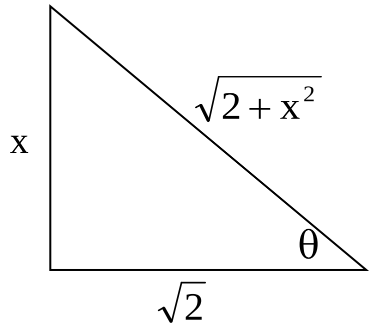
$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \cosh^{-1}\frac{x}{3} + c = \ln\left(\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right) + c$$

(3) $\int \frac{1}{\sqrt{2+x^2}}\, dx$

Let $x = \sqrt{2}\tan\theta \Rightarrow dx = \sqrt{2}\sec^2\theta\, d\theta$

$$\sqrt{2+x^2} = \sqrt{2+2\tan^2\theta} = \sqrt{2\left(1+\tan^2\theta\right)} = \sqrt{2}\sec\theta$$

$$\therefore \int \frac{dx}{\sqrt{2+x^2}} = \int \frac{\sqrt{2}\sec^2\theta}{\sqrt{2}\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln\left|\sec\theta + \tan\theta\right| + c$$



$\because x = \sqrt{2}\tan\theta \Rightarrow \tan\theta = \dfrac{x}{\sqrt{2}}$ and $\sec\theta = \dfrac{\sqrt{2+x^2}}{\sqrt{2}}$

$$\therefore \int \frac{dx}{\sqrt{2+x^2}} = \ln\left|\frac{\sqrt{2+x^2}}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right| + c$$

(4) $\int \sqrt{9-x^2}\, dx$

Let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$

$$\therefore \int \sqrt{9-x^2}\, dx = \int 3\cos\theta \times 3\cos\theta\, d\theta = 9\int \cos^2\theta\, d\theta = \frac{9}{2}\int \left(1+\cos 2\theta\right) d\theta$$

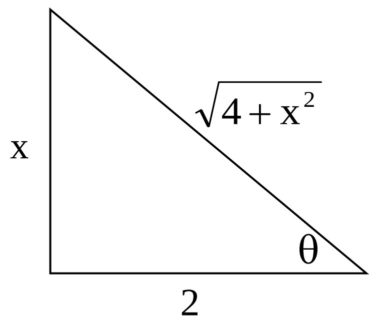
$$= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{4}\sin\left(2\sin^{-1}\frac{x}{3}\right) + c$$

(5) $\int \dfrac{dx}{\left(4+x^2\right)^{3/2}}$

Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$ and $\left(4+x^2\right)^{3/2} = \left(4+4\tan^2\theta\right)^{3/2}$

$$\therefore \left(4+x^2\right)^{3/2} = \left(4\sec^2\theta\right)^{3/2} = 4^{3/2}\sec^3\theta = 8\sec^3\theta$$

$$\therefore \int \frac{dx}{\left(4+x^2\right)^{3/2}} = \int \frac{2\sec^2\theta\, d\theta}{8\sec^3\theta} = \int \frac{d\theta}{4\sec\theta} = \frac{1}{4}\int \cos\theta\, d\theta = \frac{1}{4}\sin\theta + c$$



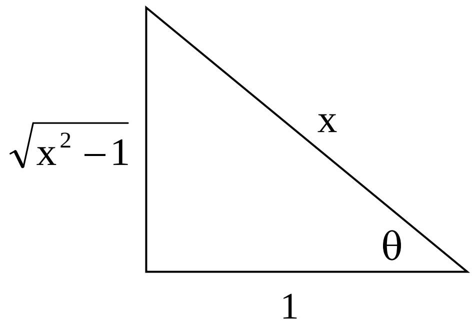
$\because x = 2\tan\theta \Rightarrow \tan\theta = \dfrac{x}{2}$ and $\sin\theta = \dfrac{x}{\sqrt{4+x^2}}$

$$\therefore \int \frac{dx}{\left(4+x^2\right)^{3/2}} = \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + c$$

(6) $\int \dfrac{x^2\, dx}{\sqrt{x^2-1}}$

Let $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta\, d\theta$ and $\sqrt{x^2 - 1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$\therefore \int \frac{x^2\, dx}{\sqrt{x^2 - 1}} = \int \frac{\sec^2\theta}{\tan\theta}\sec\theta\tan\theta\, d\theta = \int \sec^3\theta\, d\theta$$



$$\therefore \int \frac{x^2\, dx}{\sqrt{x^2 - 1}} = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$

$$\because x = \sec\theta \Rightarrow \sec\theta = \frac{x}{1} \Rightarrow \tan\theta = \sqrt{x^2 - 1}$$

$$\therefore \int \frac{x^2\, dx}{\sqrt{x^2 - 1}} = \frac{x\sqrt{x^2 - 1}}{2} + \frac{1}{2}\ln\left|x + \sqrt{x^2 - 1}\right| + c$$

4- *Integrals Involving* $ax^2 + bx + c$

Ex4: Find:

$$(1)\int \frac{dx}{x^2 + 2x + 10}$$

$$\because x^2 + 2x + 10 = x^2 + 2x + 1 - 1 + 10 = (x+1)^2 - 1 + 10 = 9 + (x+1)^2$$

$$\therefore \int \frac{dx}{x^2 + 2x + 10} = \int \frac{dx}{9 + (x+1)^2}$$

Let $u = x + 1 \Rightarrow du = dx$

$$\therefore \int \frac{dx}{9 + (x+1)^2} = \int \frac{du}{9 + u^2}$$

Let $u = 3\tan\theta \Rightarrow du = 3\sec^2\theta\, d\theta$ and $9 + u^2 = 9\sec^2\theta$

$$\therefore \int \frac{du}{9 + u^2} = \int \frac{3\sec^2\theta\, d\theta}{9\sec^2\theta} = \frac{1}{3}\int d\theta = \frac{1}{3}\theta + c = \frac{1}{3}\tan^{-1}\frac{u}{3} + c = \frac{1}{3}\tan^{-1}\frac{x+1}{3} + c$$

$$\therefore \int \frac{dx}{x^2 + 2x + 10} = \frac{1}{3}\tan^{-1}\left[\frac{x+1}{3}\right] + c$$

$$(2)\int \frac{dx}{\sqrt{2x - x^2}}$$

$$\because 2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1 - 1) = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$$

$$\therefore \int \frac{dx}{\sqrt{2x - x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + c$$

Or $x - 1 = \sin\theta \Rightarrow dx = \cos\theta\, d\theta \Rightarrow \sqrt{1-(x-1)^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

$$\therefore \int \frac{dx}{\sqrt{1-(x-1)^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta + c = \sin^{-1}(x-1) + c$$

$$(3)\int \frac{dx}{(2x+1)\sqrt{x^2+x}}$$

$$\because\ x^2 + x = x^2 + x + \frac{1}{4} - \frac{1}{4} = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}(2x+1)^2 - \frac{1}{4}$$

$$\therefore \int \frac{dx}{(2x+1)\sqrt{x^2+x}} = \int \frac{dx}{(2x+1)\sqrt{0.25(2x+1)^2 - 0.25}} = \int \frac{dx}{0.5(2x+1)\sqrt{(2x+1)^2 - 1}}$$

$$= \int \frac{2\,dx}{(2x+1)\sqrt{(2x+1)^2-1}} = \sec^{-1}(2x+1) + c$$

Or $2x + 1 = \sec\theta \Rightarrow 2\,dx = \sec\theta\tan\theta\, d\theta$

$$\therefore \sqrt{(2x+1)^2 - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

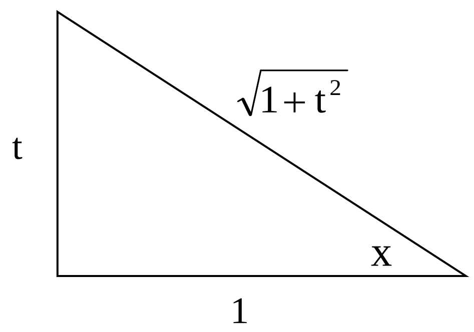
$$\therefore \int \frac{2\,dx}{(2x+1)\sqrt{(2x+1)^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta} = \int d\theta = \theta + c = \sec^{-1}(2x+1) + c$$

5- *Integrals Of The Form* $\int \frac{dx}{a\sin^2 x + b\cos^2 x + c}$

The key to the method is to substitute $t = \tan x$ in the integral.

$\because\ \tan x = t$

$$\therefore \sin x = \frac{t}{\sqrt{1+t^2}} \text{ and } \cos x = \frac{1}{\sqrt{1+t^2}}$$



$$\therefore x = \tan^{-1} t \Rightarrow \frac{dx}{dt} = \frac{1}{1+t^2} \Rightarrow dx = \frac{dt}{1+t^2}$$

<u>Ex5:</u> Find:

$$(1)\int \frac{dx}{1+\cos^2 x}$$

Let $\tan x = t \Rightarrow dx = \dfrac{dt}{1+t^2}$, $\cos x = \dfrac{1}{\sqrt{1+t^2}}$

Then $1+\cos^2 x = 1+\dfrac{1}{1+t^2} = \dfrac{1+t^2+1}{1+t^2} = \dfrac{2+t^2}{1+t^2}$

$$\therefore \int \frac{dx}{1+\cos^2 x} = \int \frac{1+t^2}{2+t^2}\frac{dt}{1+t^2} = \int \frac{dt}{2+t^2}$$

Let $t = \sqrt{2}\tan\theta \Rightarrow dt = \sqrt{2}\sec^2\theta\, d\theta \Rightarrow 2+t^2 = 2+2\tan^2\theta = 2\sec^2\theta$

$$\therefore \int \frac{dt}{2+t^2} = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{2\sec^2\theta} = \frac{1}{\sqrt{2}}\int d\theta = \frac{1}{\sqrt{2}}\theta + c = \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{t}{\sqrt{2}}\right] + c$$

$$\therefore \int \frac{dx}{1+\cos^2 x} = \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\tan x}{\sqrt{2}}\right] + c$$

$$(2)\int \frac{dx}{\sin^2 x - 4\cos^2 x}$$

Let $t = \tan x \Rightarrow dx = \dfrac{dt}{1+t^2}$, $\cos x = \dfrac{1}{\sqrt{1+t^2}}$, $\sin x = \dfrac{t}{\sqrt{1+t^2}}$

$$\therefore \sin^2 x - 4\cos^2 x = \frac{t^2}{1+t^2} - \frac{4}{1+t^2} = \frac{t^2-4}{1+t^2} \Rightarrow \frac{1}{\sin^2 x - 4\cos^2 x} = \frac{1+t^2}{t^2-4}$$

$$\therefore \int \frac{1}{\sin^2 x - 4\cos^2 x}dx = \int \frac{1+t^2}{t^2-4}\times\frac{dt}{1+t^2} = \int \frac{dt}{t^2-4} = \int \frac{1}{(t-2)(t+2)}dt$$

$$\therefore \frac{1}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2} \times (t-2)(t+2)$$

$$\therefore 1 = A(t+2) + B(t-2) = At + 2A + Bt - 2B$$

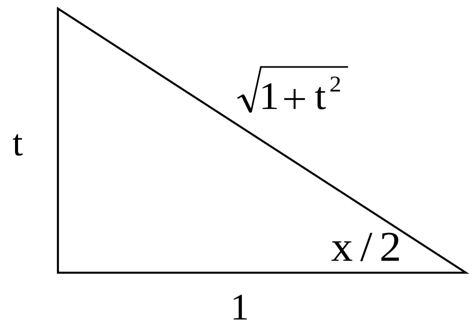
$$\therefore A+B = 0 \text{ and } 2A - 2B = 1 \Rightarrow A = \frac{1}{4} \text{ and } B = \frac{-1}{4}$$

$$\therefore \int \frac{1}{(t-2)(t+2)}dt = \frac{1}{4}\int \frac{1}{t-2}dt - \frac{1}{4}\int \frac{1}{t+2}dt = \frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2| + c = \frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + c$$

$$\therefore \int \frac{1}{\sin^2 x - 4\cos^2 x}dx = \frac{1}{4}\ln\left|\frac{\tan x - 2}{\tan x + 2}\right| + c$$

6- *Integrals Of The Form* $\int \dfrac{dx}{a\sin x + b\cos x + c}$

The key this time is to substitute $t = \tan\frac{x}{2}$.



$$\because \tan\frac{x}{2} = t \Rightarrow \frac{x}{2} = \tan^{-1} t \Rightarrow x = 2\tan^{-1} t$$

$$\therefore \frac{dx}{dt} = \frac{2}{1+t^2} \Rightarrow dx = \frac{2\,dt}{1+t^2}$$

$$\because \sin x = \sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$$\because \cos x = \cos 2\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

Ex6: Find:

$$(1)\int \frac{dx}{5+4\cos x}$$

Let $t = \tan(x/2)$

$$\therefore dx = \frac{2\,dt}{1+t^2} \text{ and } \cos x = \frac{1-t^2}{1+t^2}$$

$$\therefore 5+4\cos x = 5+4\frac{1-t^2}{1+t^2} = 5+\frac{4-4t^2}{1+t^2} = \frac{5+5t^2+4-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2}$$

$$\therefore \int \frac{dx}{5+4\cos x} = \int \frac{1+t^2}{9+t^2}\frac{2\,dt}{1+t^2} = \int \frac{2\,dt}{9+t^2}$$

Let $t = 3\tan\theta \Rightarrow dt = 3\sec^2\theta\, d\theta \Rightarrow 9+t^2 = 9+9\tan^2\theta = 9\sec^2\theta$

$$\therefore \int \frac{2\,dt}{9+t^2} = \int \frac{2\times 3\sec^2\theta\, d\theta}{9\sec^2\theta} = \frac{2}{3}\int d\theta = \frac{2}{3}\theta + c = \frac{2}{3}\tan^{-1}\left[\frac{t}{3}\right] + c$$

$$\therefore \int \frac{dx}{5+4\cos x} = \frac{2}{3}\tan^{-1}\left[\frac{\tan(x/2)}{3}\right] + c$$

$$(2)\int \frac{1}{1+\sin x - \cos x}\,dx$$

Let $t = \tan(x/2) \Rightarrow dx = \frac{2\,dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore 1+\sin x-\cos x=1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}=\frac{1+t^2+2t-1+t^2}{1+t^2}=\frac{2t^2+2t}{1+t^2}$$

$$\therefore \int\frac{1}{1+\sin x-\cos x}dx=\int\frac{1+t^2}{2\left(t^2+t\right)}\frac{2\,dt}{1+t^2}=\int\frac{1}{t^2+t}dt=\int\frac{1}{t\left(t+1\right)}dt$$

$$\because \frac{1}{t\left(t+1\right)}=\frac{A}{t}+\frac{B}{t+1}$$

$\therefore$ A = 1 and B = −1

$$\therefore \int\frac{1}{1+\sin x-\cos x}dx=\int\frac{1}{t}dt-\int\frac{1}{t+1}dt=\ln|t|-\ln|t+1|+c=\ln\left|\frac{t}{t+1}\right|+c$$

$$\therefore \int\frac{1}{1+\sin x-\cos x}dx=\ln\left|\frac{\tan(x/2)}{\tan(x/2)+1}\right|+c$$

Home works (3)

Q1: Determine the followings:

(1) $\int x^3 \ln x\,dx$ (2) $\int x^2 \cos x\,dx$ (3) $\int x e^{5x}\,dx$

(4) $\int e^{2x} \sin x\,dx$ (5) $\int \sin^{-1} x\,dx$ (6) $\int \csc^3 x\,dx$

(7) $\int x \sec^2 x\,dx$

Q2: Evaluate the following integrals:

(1) $\int \frac{4x+1}{(x+1)(x-2)}\,dx$ (2) $\int \frac{8}{(x-1)(3x+2)}\,dx$ (3) $\int \frac{2-x}{x(x^2+1)}\,dx$

(4) $\int \frac{8x^2+3}{(4x+1)^2}\,dx$ (5) $\int \frac{x^3+4}{x(x+1)^2}\,dx$

Q3: Estimate the following integrals:

(1) $\int \frac{4}{\sqrt{9+x^2}}\,dx$ (2) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ (3) $\int \frac{\sqrt{x^2-4}}{x}\,dx$

(4) $\int \sqrt{1+x^2}\,dx$ (5) $\int \frac{1}{(a^2-x^2)^{3/2}}\,dx$

Q4: Calculate the following integrals:

(1) $\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$ (2) $\int \sqrt{2x-x^2}\,dx$ (3) $\int \frac{1}{\sqrt{x^2-2x+2}}\,dx$

Q5: Achieve the following integrals:

(1) $\int \frac{dx}{5-\sin^2 x}$ (2) $\int \frac{dx}{9\cos^2 x-\sin^2 x}$

Q6: Evaluate the following integrals:

(1) $\int \frac{dx}{1+\sin x}$ (2) $\int \frac{dx}{1-3\sin x+\cos x}$