Alternating Series

A series in which the terms are alternately positive and negative.

Example

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{(-1)^{n+1}}{n}+\ldots$$

$$-2+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n}4}{2^{n}}+\ldots$$

$$1-2+3-4+5-6+\ldots+(-1)^{n+1}n+\ldots$$

The Convergence Test of Alternating Series

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1-u_2+u_3-u_4+\ldots$$

converges if all three of the following conditions are satisfied:

1) The u_n's are all positive.
2) $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3) $u_n \to 0$.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

satisfies the three requirements of convergence; it therefore converges.

Absolute Convergence

A series $\sum a_n$ ***converges absolutely*** (is ***absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges, i.e.,

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Example

The geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ converges absolutely because the corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges .

Conditional Convergence

A series that converges but does not converge absolutely ***converges conditionally***.

Example

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ does not converge absolutely. The corresponding series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the divergent harmonic series.

Power Series

❖ A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

❖ A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots$$

in which the center a and the coefficients $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

Example

The series $\sum_{n=0}^{\infty} x^n$ is a geometric series with first term 1 and ratio x. It converges to

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \qquad \text{for } |x| < 1$$

Convergence of Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.

The test of power series is done using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \rho \begin{cases} < 1 & Conv. \\ > 1 & Div. \\ = 1 & Fails \end{cases}$$

Notes:

- Use the Ratio Test to find the interval where the series converges absolutely.
- If the interval of absolute convergence is finite, test the convergence or divergence at each endpoint. Use the integral test or the Alternating Series Test for endpoints.

❖ If the interval of absolute convergence is $|x-a|<R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the nth-term does not approach zero for those values of x.

Example

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots,$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,$

(d) $\sum_{n=0}^{\infty}n!x^n=1+x+2!x^2+3!x^3+\ldots$

Solution

(a) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right|=\frac{n}{n+1}|x|\to|x|.$

The series converges absolutely for $|x|<1$. It diverges if $|x|>1$ because the nth-term does not converge to zero. At $x=1$, we get the alternating harmonic series $1-1/2+1/3-1/4+\ldots$, which converges. At $x=-1$, we get $-1-1/2-1/3-1/4-\ldots$, the negative of the harmonic series; it diverges. So, the series converges for $-1<x\le 1$ and diverges elsewhere.

(b) $\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{x^{2n+1}}{2n+1}\times\frac{2n-1}{x^{2n-1}}\right|=\frac{2n-1}{2n+1}x^2\to x^2.$

The series converges absolutely for $x^2<1$. It diverges for $x^2>1$ because the nth-term does not converge to zero. At $x=1$, the series becomes $1-1/3+1/5-1/7+\ldots$, which converges because it satisfies the three conditions of convergence of alternating series. It also converges at $x=-1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x=-1$ is the negative of the value at $x=1$. So, the series converges for $-1 \le x \le 1$ and diverges elsewhere.

(c) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \dfrac{|x|}{n+1} \to 0$ for every x.

The series converges absolutely for all x.

(d) $\left|\dfrac{u_{n+1}}{u_n}\right| = \left|\dfrac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \to \infty$ unless $x=0$.

The series diverges for all values of x except $x=0$.

Exercises on Alternating & Power Series

Which of the following series converges and which diverges?

1) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ ***Ans.*** *Converges*

2) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{10}\right)^n$ ***Ans.*** *Diverges,* $a_n \to \infty$

3) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln n}$ ***Ans.*** *Converges*

4) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln(n^2)}$ ***Ans.*** *Diverges,* $a_n \to \dfrac{1}{2}$

5) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$ ***Ans.*** *Converges*