Fourier Series

Periodic Function

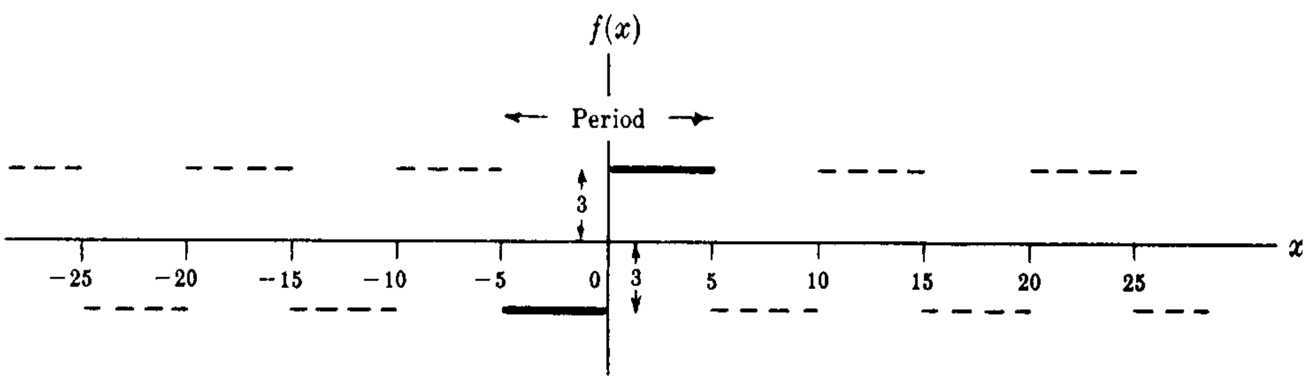
A function $f(x)$ is said to have a ***period*** T or to be ***periodic*** with period T if for all t, $f(t+T)=f(t)$, where T is a positive constant. The least value of $T>0$ is called ***the period*** of $f(t)$.

Example

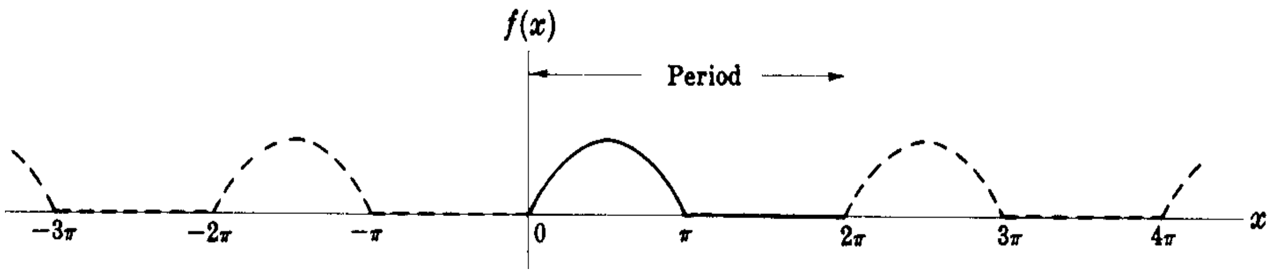
- The function $\sin(x)$ has period 2π, since $\sin(x+2\pi)=\sin(x)$.
- The period of $\sin(nx)$ or $\cos(nx)$, where n is a positive integer, is $2\pi/n$.

Example

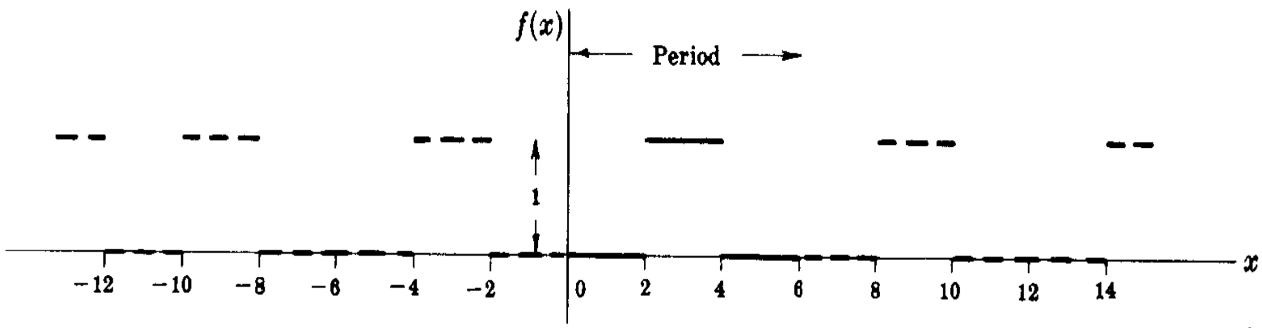
- $f(x)=\begin{cases} 3 & 0<x<5 \\ -3 & -5<x<0 \end{cases}$ $\qquad Period=10$



- $f(x)=\begin{cases} \sin(x) & 0<x<\pi \\ 0 & \pi<x<2\pi \end{cases}$ $\qquad Period=2\pi$



➢ $f(x)=\begin{cases}0 & 0<x<2\\ 1 & 2<x<4\\ 0 & 4<x<6\end{cases}$ $\qquad Period = 6$



Exercises

Find the smallest positive period of the following functions

1) $\cos(x)$ ***Ans.*** 2π

2) $\sin(x)$ ***Ans.*** 2π

3) $\cos(2x)$ ***Ans.*** π

4) $\sin(2x)$ ***Ans.*** π

5) $\cos(\pi x)$ ***Ans.*** 2

6) $\sin(\pi x)$ ***Ans.*** 2

7) $\cos(2\pi x)$ ***Ans.*** 1

8) $\sin(2\pi x)$ ***Ans.*** 1

9) $\cos(nx)$ ***Ans.*** $2\pi / n$

10) $\sin(nx)$ ***Ans.*** $2\pi / n$

11) $\cos\left(\frac{2\pi x}{k}\right)$ ***Ans.*** k

12) $\sin\left(\frac{2\pi x}{k}\right)$ ***Ans.*** k

13) $\cos\left(\frac{2\pi nx}{k}\right)$ ***Ans.*** $\frac{k}{n}$

14) $\sin\left(\frac{2\pi nx}{k}\right)$ ***Ans.*** $\frac{k}{n}$

Fourier Series

Let $f(t)$ is a periodic function with a period of T. The ***Fourier Series*** or ***Fourier Expansion*** corresponding to $f(t)$ is given by

$$f(t) = d_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)dt \qquad n = 0,1,2,\ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega_0 t)dt \qquad\qquad n = 0,1,2,...$$

with $\omega_0 = \dfrac{2\,\pi}{T}$

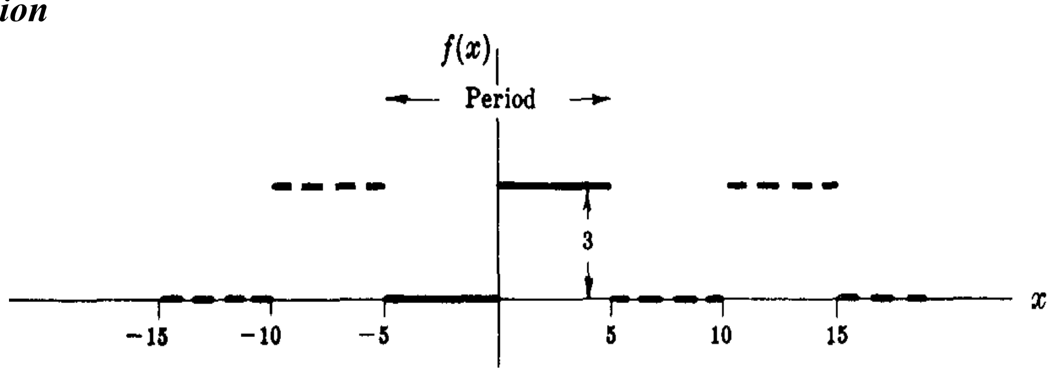
and $$d_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

Example

Find the Fourier series corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad\qquad Period = 10$$

Solution



$$T = 10\,,\ \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$d_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)dx = \frac{1}{10}\int_{-5}^{5} f(x)dx = \frac{1}{10}\int_{0}^{5} 3dx = \frac{3}{10}x\Big|_0^5 = \frac{3}{10}(5-0) = \frac{3}{2}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(n\omega_0 x)dx = \frac{2}{10}\int_{0}^{5} 3\cos(\frac{\pi}{5}nx)dx = \frac{3}{5}\times\frac{5}{n\pi}\sin(\frac{\pi}{5}nx)\Big|_0^5$$

$$= \frac{3}{n\pi}\left[\sin\left(\frac{\pi}{5}\times 5n\right) - \sin(0)\right] = \frac{3}{n\pi}\sin(n\pi) = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(n\omega_0 x)dx = \frac{2}{10}\int_{-5}^{5} f(x)\sin\left(\frac{\pi}{5}nx\right)dx = \frac{1}{5}\int_{0}^{5} 3\sin\left(\frac{\pi}{5}nx\right)dx$$

$$= \frac{3}{5}\times\frac{5}{n\pi}\cos\left(\frac{\pi}{5}nx\right)\Bigg|_{5}^{0} = \frac{3}{n\pi}\left[1 - \cos\left(\frac{\pi}{5}\times 5n\right)\right] = \frac{3}{n\pi}\left[1 - \cos(n\pi)\right]$$

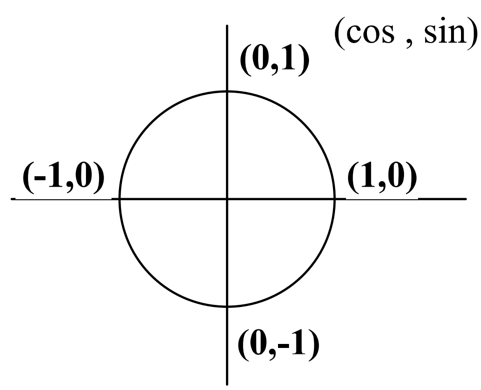
$$\cos(n\pi) = \begin{cases} +1 & n \ \ even \\ -1 & n \ \ odd \end{cases} \quad \Rightarrow \quad b_n = \begin{cases} 0 & n \ \ even \\ 6/n\pi & n \ \ odd \end{cases}$$

The corresponding Fourier series is

$$f(x) = d_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 x)$$

$$f(x) = \frac{3}{2} + \frac{6}{\pi}\left(\sin\left(\frac{\pi}{5}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{5}x\right) + \frac{1}{5}\sin(\pi x) + ...\right)$$

Notes:



- $\sin(n\pi) = 0, \qquad \sin(2n\pi) = 0$
- $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \ \ even \\ +1 & n = 1,5,9,... \\ -1 & n = 3,7,11,... \end{cases}$
- $\cos(n\pi) = \begin{cases} +1 & n \ \ even \\ -1 & n \ \ odd \end{cases}$
- $\cos(2n\pi) = 1$
- $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \ \ odd \\ +1 & n = 4,8,12,... \\ -1 & n = 2,6,10,... \end{cases}$

➢ $$\int_{-T/2}^{T/2} \sin(k\omega_0 t)dt = \int_{-T/2}^{T/2} \cos(k\omega_0 t)dt = 0 \qquad \text{if} \qquad k = 1,2,3,...$$

Proof

$$\int_{-T/2}^{T/2} \sin(k\omega_0 t)dt = -\frac{1}{k\omega_0}\cos(k\omega_0 t)\Big|_{-T/2}^{T/2}$$

$$= -\frac{1}{k\omega_0}\left(\cos\left(k\frac{2\pi}{T}\times\frac{T}{2}\right) - \cos\left(k\frac{2\pi}{T}\times\frac{-T}{2}\right)\right)$$

$$= -\frac{1}{k\omega_0}\big(\cos(k\pi) - \cos(-k\pi)\big) = 0$$

$$\int_{-T/2}^{T/2} \cos(k\omega_0 t)dt = \frac{1}{k\omega_0}\sin(k\omega_0 t)\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{k\omega_0}\left(\sin\left(k\frac{2\pi}{T}\times\frac{T}{2}\right) - \sin\left(k\frac{2\pi}{T}\times\frac{-T}{2}\right)\right)$$

$$= \frac{1}{k\omega_0}\big(\sin(k\pi) - \sin(-k\pi)\big) = 0$$

➢ $$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

where m and n assume any of the values $1,2,3,...$

Proof

Using the trigonometry $\cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$ then

If $m \neq n$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \cos((m-n)\omega_0 t) + \frac{1}{2}\int_{-T/2}^{T/2} \cos((m+n)\omega_0 t) = 0$$

Also, by using $\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$ then

If $m \neq n$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \cos((m-n)\omega_0 t) - \frac{1}{2}\int_{-T/2}^{T/2} \cos((m+n)\omega_0 t) = 0$$

If $m = n$, we have

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \big(1 + \cos(2n\omega_0 t)\big)dt$$

$$= \frac{1}{2}\int_{-T/2}^{T/2} dt + \frac{1}{2}\int_{-T/2}^{T/2} \cos(2n\omega_0 t)dt = \frac{1}{2}\left(\frac{T}{2} + \frac{T}{2}\right) = \frac{T}{2}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \big(1 - \cos(2n\omega_0 t)\big)dt$$

$$= \frac{1}{2}\int_{-T/2}^{T/2} dt - \frac{1}{2}\int_{-T/2}^{T/2} \cos(2n\omega_0 t)dt = \frac{1}{2}\left(\frac{T}{2} + \frac{T}{2}\right) = \frac{T}{2}$$

Note that if $m = n = 0$ then $\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = T$

and $\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = 0$

- $\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0$

Proof

Using the trigonometry $\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$

If $m \neq n$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \sin((m-n)\omega_0 t)dt + \frac{1}{2}\int_{-T/2}^{T/2} \sin((m+n)\omega_0 t)dt = 0$$

If $m = n$, we have

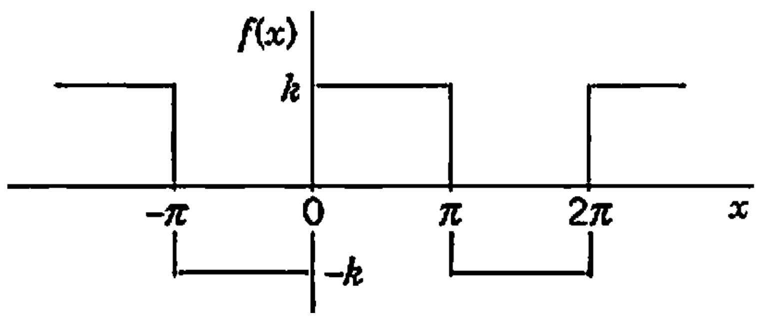
$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = \frac{1}{2}\int_{-T/2}^{T/2} \sin(2n\omega_0 t)dt = 0$$

Example

Find the Fourier series corresponding to the function

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases} \qquad Period = 2\pi$$

Solution



$$T = 2\pi \text{ , } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$d_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)dx + \int_{0}^{\pi} kdx\right] = 0$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(n\omega_0 x)dx$$

$$= \frac{2}{2\pi}\left[\int_{-\pi}^{0}(-k)\cos(nx)dx + \int_{0}^{\pi} k\cos(nx)dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{k}{n}\sin(nx)\Big|_{-\pi}^{0} + \frac{k}{n}\sin(nx)\Big|_{0}^{\pi} \right] = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(n\omega_0 x)dx$$

$$= \frac{2}{2\pi}\left[\int_{-\pi}^{0}(-k)\sin(nx)dx + \int_{0}^{\pi} k\sin(nx)dx \right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n}\cos(nx)\Big|_{-\pi}^{0} - \frac{k}{n}\cos(nx)\Big|_{0}^{\pi} \right]$$

$$= \frac{k}{n\pi}[1 - \cos(n\pi) - \cos(n\pi) + 1]$$

$$= \frac{2k}{n\pi}[1 - \cos(n\pi)]$$

$$\cos(n\pi) = \begin{cases} +1 & n \ \ even \\ -1 & n \ \ odd \end{cases} \quad \Rightarrow \quad b_n = \begin{cases} 0 & n \ \ even \\ \dfrac{4k}{n\pi} & n \ \ odd \end{cases}$$

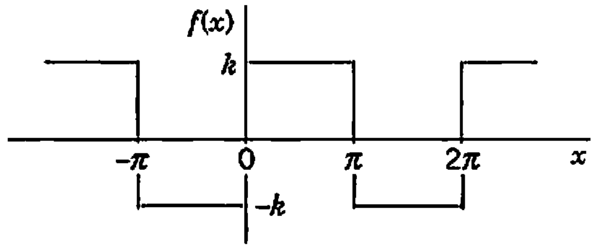
The corresponding Fourier series is

$$f(x) = d_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 x) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 x)$$

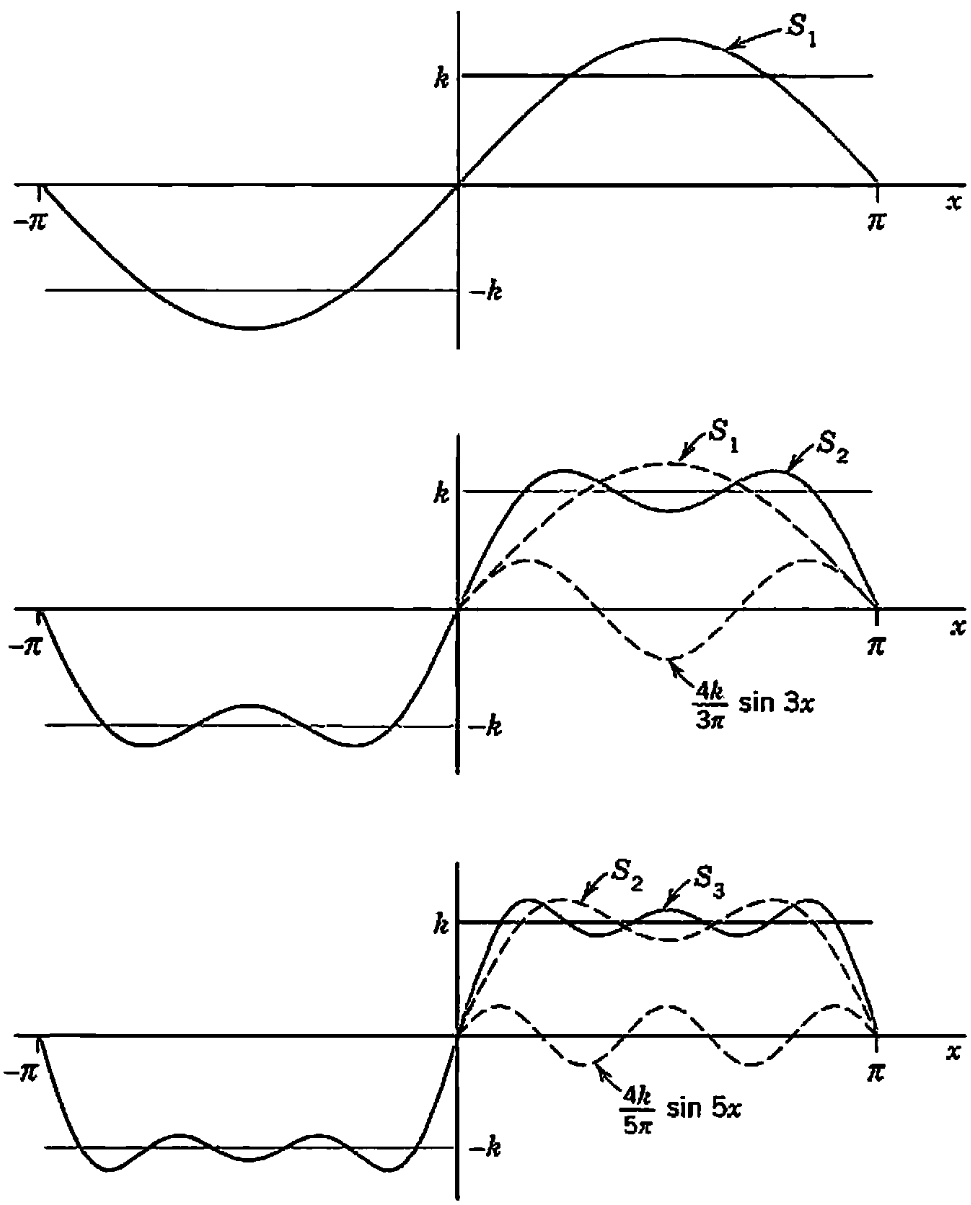
$$f(x) = \frac{4k}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(\ x) + \ ...... \right)$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin(x), \qquad S_2 = \frac{4k}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) \right]$$



(a) The given function $f(x)$ (Periodic rectangular wave)



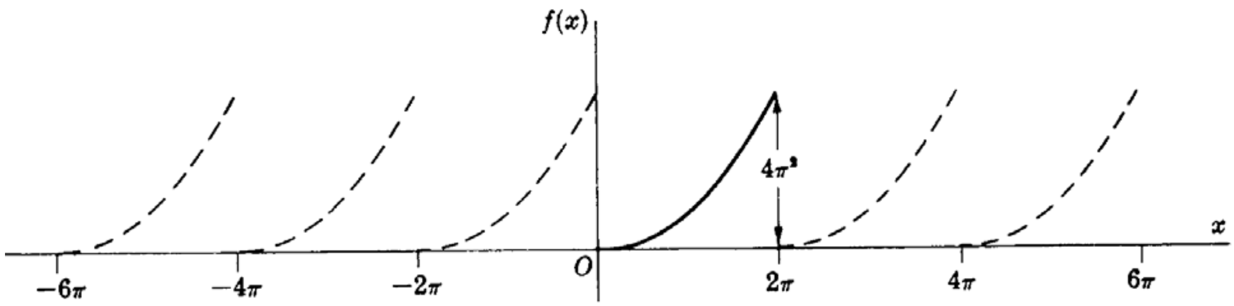
(b) The first three partial sums of the corresponding Fourier series

Example

Find the Fourier series of the periodic function

$$f(x) = x^2 \qquad\qquad 0 < x < 2\pi$$

Solution



$$T = 2\pi \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$d_0 = \frac{1}{T}\int_0^T f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left.\frac{x^3}{3}\right|_0^{2\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos(n\omega_0 x)dx = \frac{2}{2\pi}\int_0^{2\pi} x^2\cos(nx)dx$$

$$= \frac{1}{\pi}\left.\left\{(x^2)\left(\frac{\sin(nx)}{n}\right) - (2x)\left(\frac{-\cos(nx)}{n^2}\right) + 2\left(\frac{-\sin(nx)}{n^3}\right)\right\}\right|_0^{2\pi} = \frac{4}{n^2}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(n\omega_0 x)dx = \frac{2}{2\pi}\int_0^{2\pi} x^2\sin(nx)dx$$

$$= \frac{1}{\pi}\left.\left\{(x^2)\left(\frac{-\cos(nx)}{n}\right) - (2x)\left(\frac{-\sin(nx)}{n^2}\right) + 2\left(\frac{\cos(nx)}{n^3}\right)\right\}\right|_0^{2\pi} = \frac{-4\pi}{n}$$

So, $f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nx) - \sum_{n=1}^{\infty} \frac{4\pi}{n}\sin(nx)$

$$f(x) = \frac{4\pi^2}{3} + 4\left(\cos(x) + \frac{1}{4}\cos(2x) + \ldots\right) - 4\pi\left(\sin(x) + \frac{1}{2}\sin(2x) + \ldots\right)$$

Exercises

Evaluate the following integrals where $n = 0,1,2,\ldots$

1) $\int_0^{\pi} \sin(nx)dx$ ***Ans.*** $\begin{cases} 0 & n \ \ even \\ 2/n & n \ \ odd \end{cases}$

2) $\int_{-\pi}^{\pi} x\sin(nx)dx$ ***Ans.*** $\begin{cases} 0 & n = 0 \\ 2\pi/n & n = 1,3,\ldots \\ -2\pi/n & n = 2,4,\ldots \end{cases}$

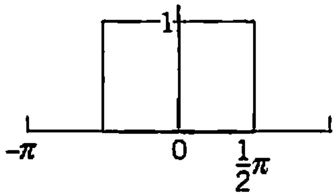
3) $\int_{-\pi/2}^{\pi/2} x \ \cos(nx)dx$ ***Ans.*** 0

4) $\int_{-\pi}^{0} e^x \sin(nx)dx$ ***Ans.*** $\frac{n\left((-1)^n e^{-\pi} - 1\right)}{\left(1 + n^2\right)}$

5) $\int_{-\pi}^{\pi} x^2 \cos(nx)dx$ ***Ans.*** $\begin{cases} 2\pi^3/3 & n = 0 \\ (-1)^n 4\pi/n^2 & n = 1,2,\ldots \end{cases}$

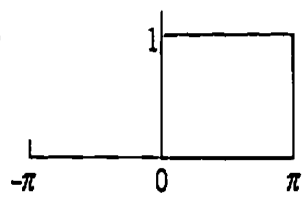
Find the Fourier Series for the following periodic functions

1)



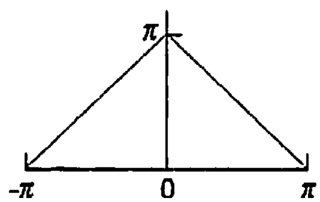
Ans. $$\frac{1}{2}+\frac{2}{\pi}\left(\cos(x)-\frac{1}{3}\cos(3x)+\frac{1}{5}\cos(5x)-\ldots\right)$$

2)



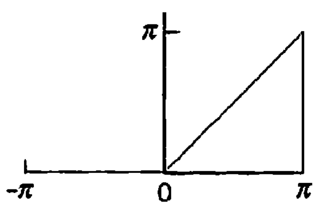
Ans. $$\frac{1}{2}+\frac{2}{\pi}\left(\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\ldots\right)$$

3)



Ans. $$\frac{\pi}{2}+\frac{4}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+\ldots\right)$$

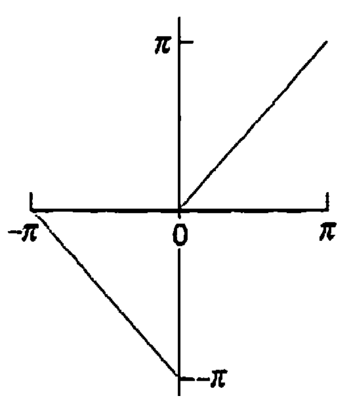
4)



Ans. $$\frac{\pi}{4}-\frac{2}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+\ldots\right)$$

$$+\sin(x)-\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)-\ldots$$

5)



Ans. $$-\frac{4}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+\ldots\right)$$

$$+2\left(\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\ldots\right)$$

Find the Fourier Series for the following periodic functions

1) $f(x)=\begin{cases}1 & -\pi/2<x<\pi/2\\ -1 & \pi/2<x<3\pi/2\end{cases}$ ***Ans.*** $\frac{4}{\pi}\left(\cos(x)-\frac{1}{3}\cos(3x)+\frac{1}{5}\cos(5x)-\ldots\right)$

2) $f(x)=x \quad -\pi<x<\pi$ ***Ans.*** $2\left(\sin(x)-\frac{1}{2}\sin(2x)+\frac{1}{3}\sin(3x)-\ldots\right)$

3) $f(x)=x^2 \quad -\pi<x<\pi$ ***Ans.*** $\frac{\pi^2}{3}-4\left(\cos(x)-\frac{1}{4}\cos(2x)+\frac{1}{9}\cos(3x)-\ldots\right)$

4) $f(x)=\begin{cases}\pi+x & -\pi<x<0\\ \pi-x & 0<x<\pi\end{cases}$ ***Ans.*** $\frac{\pi}{2}+\frac{4}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+\ldots\right)$

5) $f(x)=\begin{cases}x & -\pi/2<x<\pi/2\\ \pi-x & \pi/2<x<3\pi/2\end{cases}$ ***Ans.*** $\frac{4}{\pi}\left(\sin(x)-\frac{1}{9}\sin(3x)+\frac{1}{25}\sin(5x)-\ldots\right)$

6) $f(x)=\begin{cases}x^2 & -\pi/2<x<\pi/2\\ \pi^2/4 & \pi/2<x<3\pi/2\end{cases}$ ***Ans.*** $\frac{\pi^2}{6}-\frac{4}{\pi}\cos(x)-\frac{1}{2}\cos(2x)+\frac{4}{27\pi}\cos(3x)$
$+\frac{1}{8}\cos(4x)-\ldots$

7) $f(x)=\begin{cases}-1 & -1<x<0\\ 1 & 0<x<1\end{cases}$ ***Ans.*** $\frac{4}{\pi}\left(\sin(\pi x)+\frac{1}{3}\sin(3\pi x)+\frac{1}{5}\sin(5\pi x)+\ldots\right)$

8) $f(x)=\begin{cases}-1 & -2<x<0\\ 1 & 0<x<2\end{cases}$ ***Ans.*** $\frac{4}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)+\ldots\right)$

9) $f(x)=\begin{cases}0 & -2<x<0\\ 1 & 0<x<2\end{cases}$ ***Ans.*** $\frac{1}{2}+\frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)+\ldots\right)$

10) $f(x)=x^2 \quad -1<x<1$ ***Ans.*** $\frac{1}{3}-\frac{4}{\pi^2}\left(\cos(\pi x)-\frac{1}{4}\cos(2\pi x)+\frac{1}{9}\cos(3\pi x)-\ldots\right)$

11) $f(x)=\begin{cases}0 & -1<x<0\\ x & 0<x<1\end{cases}$ ***Ans.*** $\frac{1}{4}-\frac{2}{\pi^2}\left(\cos(\pi x)+\frac{1}{9}\cos(3\pi x)+\ldots\right)$
$+\frac{1}{\pi}\left(\sin(\pi x)-\frac{1}{2}\sin(2\pi x)+\ldots\right)$

12) $f(x)=\sin(\pi x) \quad 0<x<1$ ***Ans.*** $\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{1}{(1)(3)}\cos(2\pi x)+\frac{1}{(3)(5)}\cos(4\pi x)+\ldots\right)$

13) $f(x)=|x| \quad -1<x<1$ ***Ans.*** $\frac{1}{2}-\frac{4}{\pi^2}\left(\cos(\pi x)+\frac{1}{9}\cos(3\pi x)+\frac{1}{25}\cos(5\pi x)+\ldots\right)$

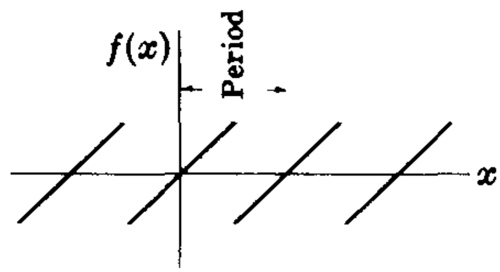
14) $f(x)=1-x^2 \quad -1<x<1$ ***Ans.*** $\frac{2}{3}+\frac{4}{\pi^2}\left(\cos(\pi x)-\frac{1}{4}\cos(2\pi x)+\frac{1}{9}\cos(3\pi x)-\ldots\right)$

15) $f(x)=\begin{cases}-1 & -1<x<0\\ 2x & 0<x<1\end{cases}$ ***Ans.*** $-\frac{4}{\pi^2}\left(\cos(\pi x)+\frac{1}{9}\cos(3\pi x)+\ldots\right)$
$+\frac{2}{\pi}\left(2\sin(\pi x)-\frac{1}{2}\sin(2\pi x)+\ldots\right)$

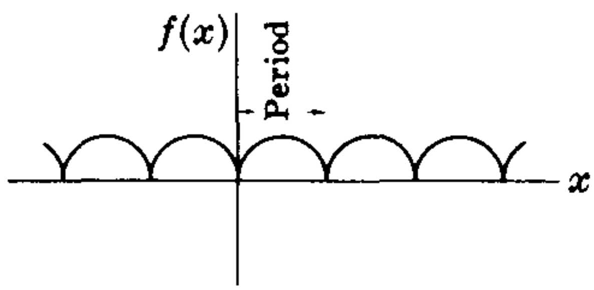
16) $f(x)=\begin{cases}-x & -1<x<0\\ x & 0<x<1\\ 1 & 1<x<3\end{cases}$ ***Ans.*** $\frac{3}{4}-\frac{4}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right)+\frac{1}{2}\cos(\pi x)+\frac{1}{9}\cos\left(\frac{3\pi x}{2}\right)\right)$
$+\frac{1}{25}\cos\left(\frac{5\pi x}{2}\right)+\frac{1}{18}\cos(3\pi x)+\ldots$

Odd and Even Functions

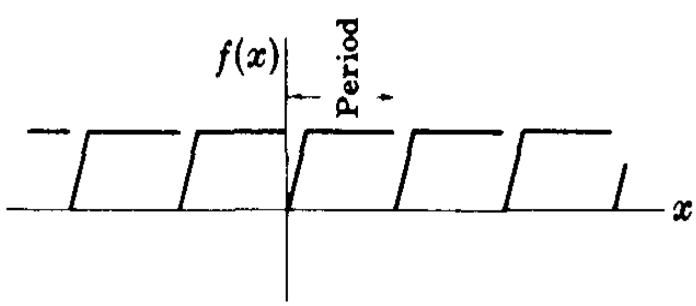
A function $f(x)$ is ***odd*** if $f(-x) = -f(x)$. Thus x^3, $x^5 - 3x^3 + 2x$, $\sin(x)$, $\tan(3x)$ are odd functions. The figure below is an example of an odd function.



A function $f(x)$ is ***even*** if $f(-x) = f(x)$. Thus x^4, $2x^6 - 4x^2 + 5$, $\cos(x)$, $e^x + e^{-x}$ are even functions. The figure below is an example of an even function.



while the figure below is neither odd nor even function.



Example

Classify each of the following functions according as they are even, odd, or neither even nor odd.

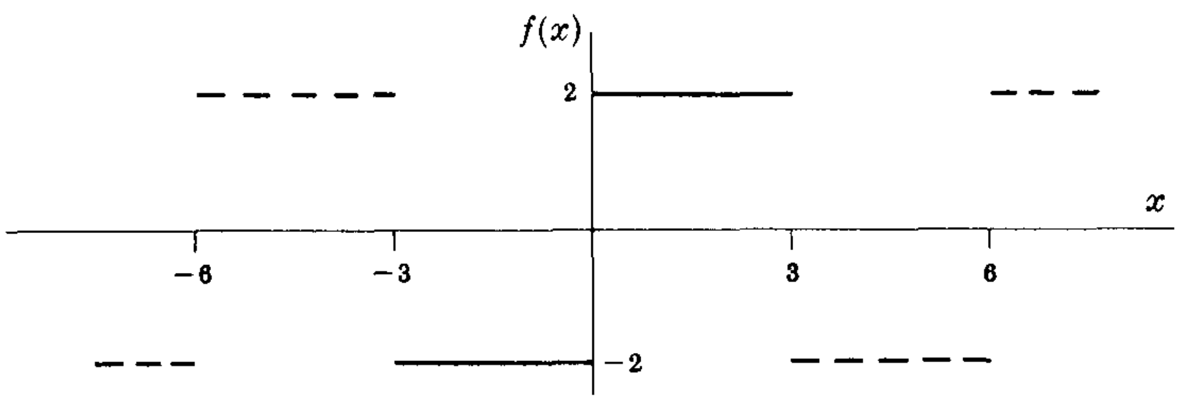
(a) $f(x)=\begin{cases} 2 & 0<x<3 \\ -2 & -3<x<0 \end{cases}$ $\quad Period=6$

(b) $f(x)=\begin{cases} \cos(x) & 0<x<\pi \\ 0 & \pi<x<2\pi \end{cases}$ $\quad Period=2\pi$

(c) $f(x)=x(10-x) \quad 0<x<10,$ $\quad Period=10$

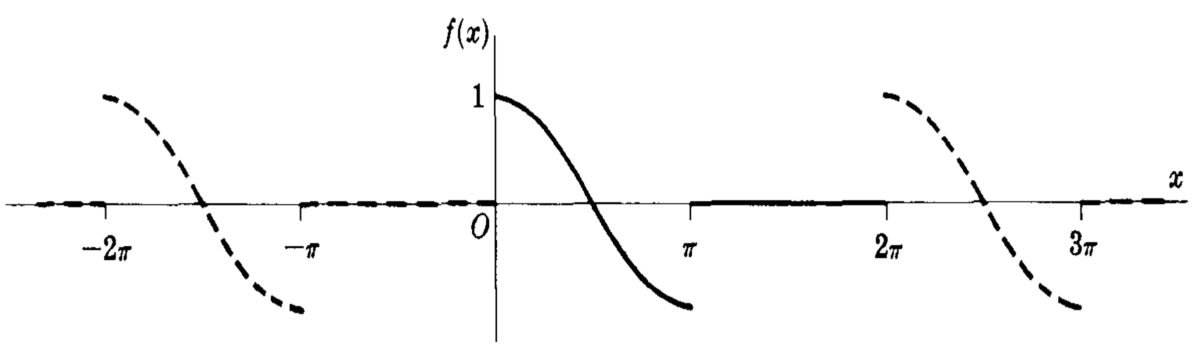
Solution

(a)



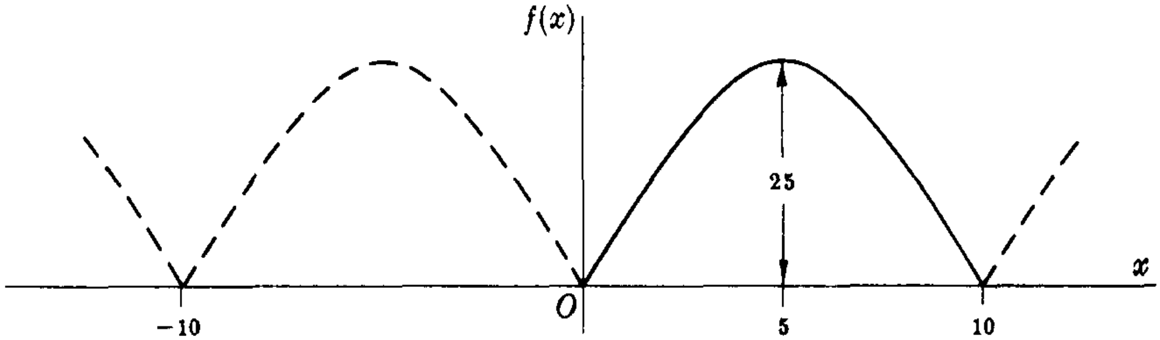
From the figure above it is seen that $f(-x)=-f(x)$, so that the function is odd.

(b)



From the above figure it is seen that the function is neither even nor odd

(c)



From the figure above it is seen that $f(-x) = f(x)$, so that the function is even.

Note:

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant) can be present.

Exercises

Are the following functions even, odd, or neither even nor odd?

1) e^x — ***Ans. Neither even nor odd***

2) e^{x^2} — ***Ans. Even***

3) $\sin(nx)$ — ***Ans. Odd***

4) $x\sin(x)$ — ***Ans. Even***

5) $\cos(x)/x$ — ***Ans. Odd***

6) $\ln(x)$ — ***Ans. Neither even nor odd***

7) $\sin(x^2)$ — ***Ans. Even***

8) $\sin^2(x)$ ***Ans. Even***

9) $|x|$ ***Ans. Even***

10) $x^2 \sin(nx)$ ***Ans. Odd***

11) $x + x^2$ ***Ans. Neither even nor odd***

12) $e^{-|x|}$ ***Ans. Even***

13) $x\cosh(x)$ ***Ans. Odd***

Are the following functions, which are assumed to be periodic, even, odd, or neither even nor odd?

1) $f(x) = x \qquad -\pi < x < \pi$ ***Ans. Odd***

2) $f(x) = x|x| \qquad -\pi < x < \pi$ ***Ans. Odd***

3) $f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ ***Ans. Neither even nor odd***

4) $f(x) = \begin{cases} x & -\pi/2 < x < \pi/2 \\ 0 & \pi/2 < x < 3\pi/2 \end{cases}$ ***Ans. Odd***

5) $f(x) = x^3 \qquad -\pi < x < \pi$ ***Ans. Odd***

6) $f(x) = e^{-4x} \qquad -\pi < x < \pi$ ***Ans. Neither even nor odd***

7) $f(x) = x|x| - x^3 \qquad -\pi < x < \pi$ ***Ans. Odd***

8) $f(x) = \begin{cases} 1/(1+x^2) & -\pi < x < 0 \\ -1/(1+x^2) & 0 < x < \pi \end{cases}$ ***Ans. Odd***

Find the Fourier Series for the following periodic functions (Even & Odd)

1) $f(x)=\begin{cases}1 & -\pi/2<x<\pi/2\\ 0 & \pi/2<x<3\pi/2\end{cases}$ ***Ans.*** $\frac{1}{2}+\frac{2}{\pi}\left(\cos(x)-\frac{1}{3}\cos(3x)+\frac{1}{5}\cos(5x)-...\right)$

2) $f(x)=\begin{cases}x & -\pi/2<x<\pi/2\\ \pi-x & \pi/2<x<3\pi/2\end{cases}$ ***Ans.*** $\frac{4}{\pi}\left(\sin(x)-\frac{1}{9}\sin(3x)+\frac{1}{25}\sin(5x)-...\right)$

3) $f(x)=\begin{cases}-x & -\pi<x<0\\ x & 0<x<\pi\end{cases}$ ***Ans.*** $\frac{\pi}{2}-\frac{4}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+...\right)$

4) $f(x)=\frac{x^2}{4} \quad -\pi<x<\pi$ ***Ans.*** $\frac{\pi^2}{12}-\cos(x)+\frac{1}{4}\cos(2x)-\frac{1}{9}\cos(3x)+...$

5) $f(x)=\pi-|x| \quad -\pi<x<\pi$ ***Ans.*** $\frac{\pi}{2}+\frac{4}{\pi}\left(\cos(x)+\frac{1}{9}\cos(3x)+\frac{1}{25}\cos(5x)+...\right)$

6) $f(x)=\begin{cases}2 & -2<x<0\\ 0 & 0<x<2\end{cases}$ ***Ans.*** $1-\frac{4}{\pi}\left(\sin\left(\frac{\pi x}{2}\right)+\frac{1}{3}\sin\left(\frac{3\pi x}{2}\right)+\frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)+...\right)$

Half Range Fourier Sine or Cosine Series

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0,T/2)$ (which is half of the interval $(0,T)$, thus accounting for the name ***half range***) and then the function is specified as ***odd*** or ***even***, so that it is clearly defined in the other half of the interval. In such case, we have for ***odd*** functions (***Sine Series***)

$$d_0 = 0, \qquad a_n = 0, \qquad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt$$

while for ***even*** functions (***Cosine Series***)

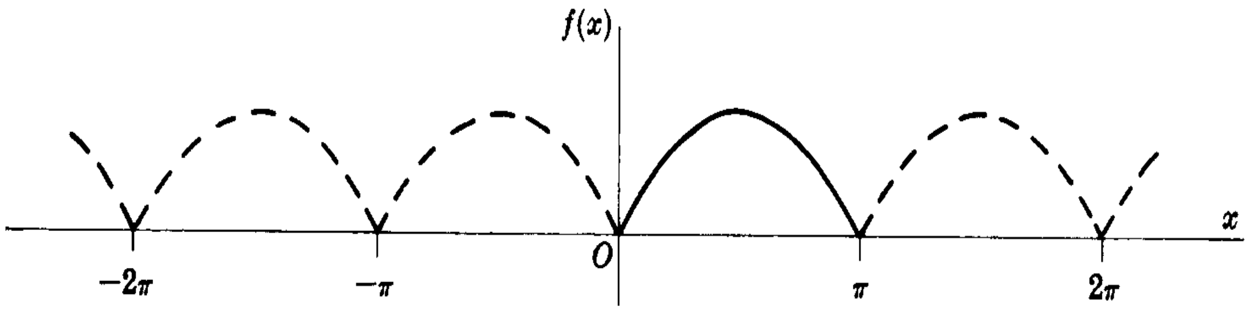
$$d_0 = \frac{2}{T}\int_0^{T/2} f(t)dt, \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt, \quad b_n = 0$$

Example

Find the Fourier series for the periodic function

$$f(x) = \sin(x) \qquad 0 < x < \pi$$

Solution



$$T = \pi \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$$

Since the function is even then $b_n = 0$

$$d_0 = \frac{2}{T}\int_0^{T/2} f(x)dx = \frac{2}{\pi}\int_0^{\pi/2}\sin(x)dx = \frac{-2}{\pi}\cos(x)\Big|_0^{\pi/2} = \frac{2}{\pi}(1-0) = \frac{2}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(x)\cos(n\omega_0 x)dx = \frac{4}{\pi}\int_0^{\pi/2}\sin(x)\cos(2nx)dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\sin\big((1-2n)x\big)dx + \frac{2}{\pi}\int_0^{\pi/2}\sin\big((1+2n)x\big)dx$$

$$= \frac{2}{\pi}\left[\frac{\cos((1-2n)x)}{1-2n} + \frac{\cos((1+2n)x)}{1+2n}\right]_{\pi/2}^{0}$$

$$= \frac{2}{\pi}\left\{\frac{1}{1-2n} + \frac{1}{1+2n} - \frac{\cos\left((1-2n)\frac{\pi}{2}\right)}{1-2n} - \frac{\cos\left((1+2n)\frac{\pi}{2}\right)}{1+2n}\right\}$$

$$= \frac{2}{\pi} \times \frac{1+2n+1-2n}{1-4n^2} = \frac{4/\pi}{1-4n^2} = \frac{-4/\pi}{4n^2-1}$$

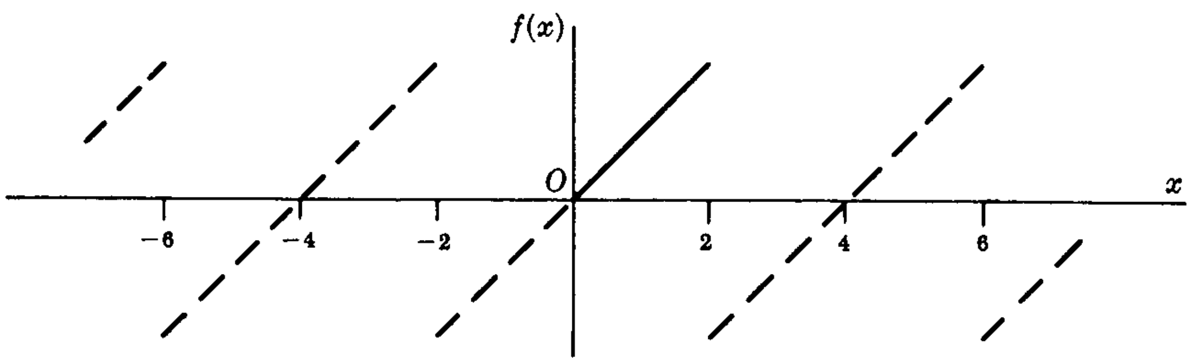
$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \dots\right)$$

Example

Expand $f(x) = x$, $0 < x < 2$ in a half range (a) sine series, (b) cosine series

Solution

(a) To get a sine series the function must be an odd function. So, we extend the given function to have an odd function. This is called the odd extension of $f(x)$.



$$T = 4 \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Since now the function is odd then $d_0 = 0$, and $a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} f(x)\sin(n\omega_0 x)dx = \frac{4}{4}\int_0^{2} x\sin\left(\frac{n\pi.x}{2}\right)dx$$

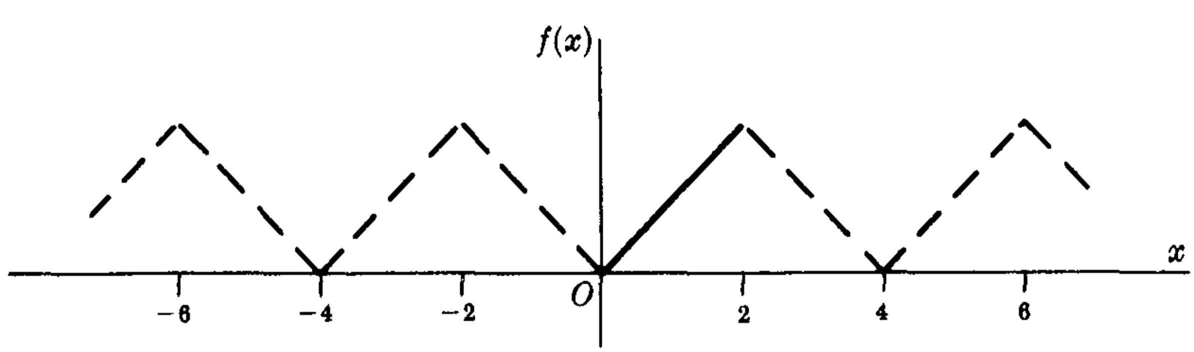
$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\left(\frac{n\pi.x}{2}\right)\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\left(\frac{n\pi.x}{2}\right)\right)\right\}\Bigg|_0^2$$

$$= \frac{-4}{n\pi}\cos(n\pi) = \begin{cases} \dfrac{-4}{n\pi} & n \;\; even \\ \\ \dfrac{4}{n\pi} & n \;\; odd \end{cases}$$

Then $f(x) = \sum_{n=1}^{\infty} \frac{-4}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi.x}{2}\right)$

$$= \frac{4}{\pi}\left[\sin\left(\frac{\pi.x}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi.x}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi.x}{2}\right) - \ldots\right]$$

(b) To get a cosine series the function must be an even function. So, we extend the given function to have an even function. This is called the even extension of $f(x)$.



$$T = 4 \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Since now the function is even then $b_n = 0$

$$d_0 = \frac{2}{T}\int_0^{T/2} f(x)dx = \frac{2}{4}\int_0^2 xdx = \frac{1}{2}\frac{x^2}{2}\bigg|_0^2 = 1$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(x)\cos(n\omega_0 x)dx = \frac{4}{4}\int_0^2 x\cos\left(\frac{n\pi.x}{2}\right)dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi.x}{2}\right)\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi.x}{2}\right)\right)\right\}\bigg|_0^2$$

$$= \frac{4}{n^2\pi^2}\left(\cos(n\pi) - 1\right) = \begin{cases} 0 & n \quad even \\ \dfrac{-8}{n^2\pi^2} & n \quad odd \end{cases}$$

Then $f(x) = d_0 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos(n\pi) - 1\right)\sin\left(\frac{n\pi.x}{2}\right)$

$$f(x) = 1 - \frac{8}{\pi^2}\left(\sin\left(\frac{\pi.x}{2}\right) + \frac{1}{9}\sin\left(\frac{3\pi.x}{2}\right) + \frac{1}{25}\sin\left(\frac{5\pi.x}{2}\right) + \ldots\right)$$

Complex Notation for Fourier Series

The Fourier series for $f(t)$ can be written in complex notation as

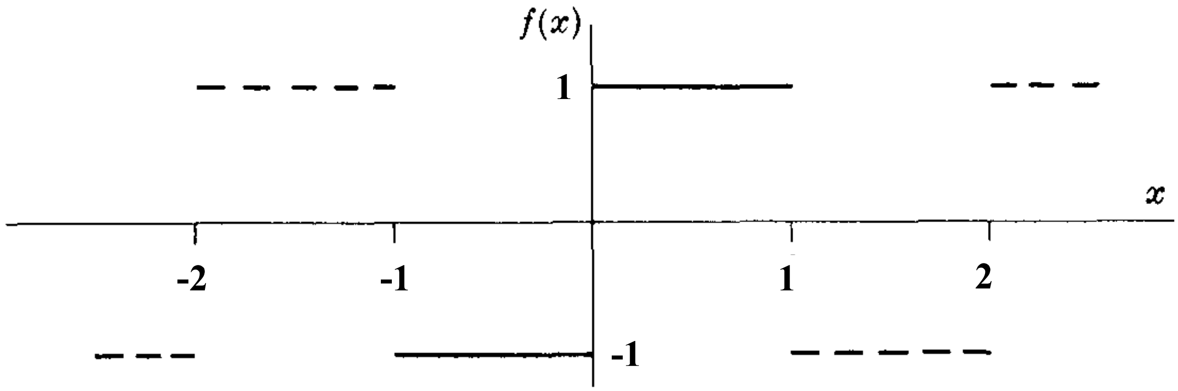
$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

Example

Write an expression for the function $f(x)$ in terms of the complex exponential Fourier series.



Solution

$$T = 2 \quad \Rightarrow \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$C_n = \frac{1}{T}\int_0^T f(x)e^{-jn\omega_0 x}dx = \frac{1}{2}\int_0^2 f(x)e^{-jn\pi x}dx = \frac{1}{2}\int_0^1 (1)e^{-jn\pi x}dx - \frac{1}{2}\int_1^2 (1)e^{-jn\pi x}dx$$

$$= \frac{1}{2jn\pi}\left(-e^{-jn\pi} + 1 + e^{-j2n\pi} - e^{-jn\pi}\right) = \frac{1}{jn\pi}\left(1 - e^{-jn\pi}\right)$$

$$C_n = \begin{cases} \dfrac{2}{jn\pi} & n \ \ odd \\ 0 & n \ \ even \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 x}$$

$$= \frac{2}{j\pi}\left(e^{j\pi x} + \frac{1}{3}e^{j3\pi x} + \frac{1}{5}e^{j5\pi x} + \ldots.. - e^{-j\pi x} - \frac{1}{3}e^{-j3\pi x} - \frac{1}{5}e^{-j5\pi x} - \ldots..\right)$$

Exercises

Find the Fourier Sine Series for the following periodic functions

1) $f(x) = x \qquad 0 < x < \pi$ ***Ans.*** $2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \ldots\right)$

2) $f(x) = x \qquad 0 < x < 1$ ***Ans.*** $\frac{2}{\pi}\left(\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \ldots\right)$

3) $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi/2 & \pi/2 < x < \pi \end{cases}$ ***Ans.*** $\left(1 + \frac{2}{\pi}\right)\sin(x) - \frac{1}{2}\sin(2x) + \left(\frac{1}{3} - \frac{2}{9\pi}\right)\sin(3x) - \frac{1}{4}\sin(4x) + \ldots$

4) $f(x) = \begin{cases} x & 0 < x < \pi/8 \\ \pi/4 - x & \pi/8 < x < \pi/4 \end{cases}$ ***Ans.*** $\frac{1}{\pi}\left(\sin(4x) - \frac{1}{9}\sin(12x) + \frac{1}{25}\sin(20x) - \frac{1}{49}\sin(28x) + \ldots\right)$

Find (a) the Fourier Cosine Series, (b) the Fourier Sine Series

1) $f(x) = 2 - x \qquad 0 < x < 2$ ***Ans.***

(a) $1 + \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{2}\right) + \ldots\right)$

(b) $\frac{4}{\pi}\left(\sin\left(\frac{\pi x}{2}\right) + \frac{1}{2}\sin(\pi x) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{4}\sin(2\pi x)\ldots\right)$

2) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$ ***Ans.***

(a) $\frac{3}{2} - \frac{2}{\pi}\left(\cos\left(\frac{\pi x}{2}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{2}\right) - \ldots\right)$

(b) $\frac{6}{\pi}\left(\sin\left(\frac{\pi x}{2}\right) - \frac{1}{3}\sin(\pi x) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right)\ldots\right)$

3) $f(x) = x \qquad 0 < x < L$ ***Ans.***

(a) $\frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\left(\frac{\pi x}{L}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{L}\right) + \ldots\right)$

(b) $\frac{2L}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \ldots\right)$

4) $f(x) = \pi - x \qquad 0 < x < \pi$ ***Ans.***

(a) $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \ldots\right)$

(b) $2\left(\sin(x) + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \ldots\right)$

Find the Complex Form of the Fourier Series for the following periodic functions

1) $f(x) = x \qquad -\pi < x < \pi$ ***Ans.*** $j\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} e^{jnx}$

2) $f(x) = e^x \qquad -\pi < x < \pi$ ***Ans.*** $\frac{\sinh(\pi)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1 + jn}{1 + n^2} e^{jnx}$

3) $f(x) = x^2 \qquad -\pi < x < \pi$ ***Ans.*** $\frac{\pi^2}{3} + 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n^2} e^{jnx}$

4) $f(x) = x \qquad 0 < x < 2\pi$ ***Ans.*** $\pi + j\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{jnx}$