Sequences and Series

Sequences of Numbers

A ***sequence*** of numbers is a function whose domain is the set of positive integers.

Example

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = n-1$

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}, \quad . \quad . \quad .$ for a sequence whose defining rule is $a_n = \frac{1}{n}$

The index n is the ***domain*** of the sequence. While the numbers in the ***range*** of the sequence are called the ***terms*** of the sequence, and the number a_n being called the ***n^{th}-term***, or ***the term with index*** n.

Example $\quad a_n = \frac{n+1}{n}$ then the terms are

$$\begin{array}{ccccc} 1^{st} \ term & 2^{nd} \ term & 3^{rd} \ term & & n^{th} \ term \\ a_1 = 2, & a_2 = \frac{3}{2}, & a_3 = \frac{4}{3}, & . \quad . \quad . & a_n = \frac{n+1}{n}, \quad . \quad . \quad . \end{array}$$

and we use the notation $\{a_n\}$ as the sequence a_n.

Example

Find the first five terms of the following:

(a) $\left\{\frac{2n-1}{3n+2}\right\}$, (b) $\left\{\frac{1-(-1)^n}{n^3}\right\}$, (c) $\left\{(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\frac{1}{5}, \quad \frac{3}{8}, \quad \frac{5}{11}, \quad \frac{7}{14}, \quad \frac{9}{17}$ **(b)** $2, \quad 0, \quad \frac{2}{27}, \quad 0, \quad \frac{2}{125}$

(c) $x, \quad \frac{-x^3}{3!}, \quad \frac{x^5}{5!}, \quad \frac{-x^7}{7!}, \quad \frac{x^9}{9!}$

Example

Find the n[th]-term of the following:

(a) $0, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4},$ (b) $0, \ \frac{\ln 2}{2}, \ \frac{\ln 3}{3}, \ \frac{\ln 4}{4},$ (c) $0, \ \frac{1}{4}, \ \frac{2}{9}, \ \frac{3}{16},$

(d) $2, \ 1, \ \frac{2^3}{3^2}, \ \frac{2^4}{4^2}, \ \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n},$ **(b)** $a_n = \frac{\ln n}{n},$ **(c)** $a_n = \frac{n-1}{n^2},$ **(d)** $a_n = \frac{2^n}{n^2}$

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

From that we can say that

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty} \{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty} \{ka_n\} = kA$

iii) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$, provided $B \neq 0$ and b_n is never 0

Example

Test the convergence of the following:

(a) $\left\{\frac{1}{n}\right\}$, (b) $\{1+(-1)^n\}$, (c) $\{n^2\}$, (d) $\{\sqrt{n+1}-\sqrt{n}\}$,

(e) $\left\{\frac{3n^2-5n}{5n^2+2n+6}\right\}$, (f) $\left\{\frac{3n^2-4n}{2n-1}\right\}$, (g) $\left\{\left(\frac{2n-3}{3n-7}\right)^4\right\}$, (h) $\left\{\frac{2n^5-4n^2}{3n^7+n^2-10}\right\}$,

(i) $\left\{\frac{2^n}{5n}\right\}$, (j) $\left\{\frac{\ln n}{e^n}\right\}$

Solution

(a) $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ ***(Conv.)***

(b) $\lim_{n\to\infty}(1+(-1)^n) = 1 + \lim_{n\to\infty}(-1)^n = \begin{cases} 0 & n \ \ odd \\ 2 & n \ \ even \end{cases}$ ***(Div.)***

(c) $\lim_{n\to\infty}(n^2) = \infty$ ***(Div.)***

(d) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty}\left((\sqrt{n+1}-\sqrt{n}) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{1}{\infty+\infty} = 0$ ***(Conv.)***

(e) $\lim_{n\to\infty}\left(\dfrac{3n^2-5n}{5n^2+2n+6}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{5n}{n^2}}{\dfrac{5n^2}{n^2}+\dfrac{2n}{n^2}+\dfrac{6}{n^2}}\right)=\dfrac{3}{5}$ ***(Conv.)***

(f) $\lim_{n\to\infty}\left(\dfrac{3n^2-4n}{2n-1}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{3n^2}{n^2}-\dfrac{4n}{n^2}}{\dfrac{2n}{n^2}-\dfrac{1}{n^2}}\right)=\dfrac{3}{0}=\infty$ ***(Div.)***

(g) $\lim_{n\to\infty}\left(\dfrac{2n-3}{3n-7}\right)^4=\left(\dfrac{2}{3}\right)^4=\dfrac{16}{81}$ ***(Conv.)***

(h) $\lim_{n\to\infty}\left(\dfrac{2n^5-4n^2}{3n^7+n^2-10}\right)=\lim_{n\to\infty}\left(\dfrac{\dfrac{2}{n^2}-\dfrac{4}{n^5}}{3+\dfrac{1}{n^5}-\dfrac{10}{n^7}}\right)=0$ ***(Conv.)***

(i) $\lim_{n\to\infty}\left(\dfrac{2^n}{5n}\right)=\lim_{n\to\infty}\left(\dfrac{2^n.\ln 2}{5}\right)=\infty$ ***(Div.)***

(j) $\lim_{n\to\infty}\left(\dfrac{\ln n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1/n}{e^n}\right)=\lim_{n\to\infty}\left(\dfrac{1}{n.e^n}\right)=\dfrac{1}{\infty}=0$ ***(Conv.)***

Example

Prove the following limits

(a) $\lim_{n\to\infty}\left(\dfrac{\ln n}{n}\right)=0$, (b) $\lim_{n\to\infty}\left(\sqrt[n]{n}\right)=1$, (c) $\lim_{n\to\infty}\left(x^{1/n}\right)=1 \quad (x>0)$,

Solution

(a) $\lim_{n\to\infty}\left(\frac{\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{1}\right) = \frac{0}{1} = 0$

(b) Let $a_n = n^{1/n}$, then $\ln a_n = \ln n^{1/n} = \frac{1}{n}\ln n \to 0$,

So, $\lim_{n\to\infty} n^{1/n} = e^{\ln a_n} \to e^0 = 1$

(c) Let $a_n = x^{1/n}$, then $\ln a_n = \ln x^{1/n} = \frac{1}{n}\ln x \to 0$,

So, $\lim_{n\to\infty} x^{1/n} = e^{\ln a_n} \to e^0 = 1$

Exercises on Sequences

Find the values of a_1, a_2, a_3 and a_4 for the following sequences

1) $a_n = \frac{1-n}{n^2}$ **2)** $a_n = \frac{1}{n!}$ **3)** $a_n = \frac{(-1)^{n+1}}{2n-1}$

4) $a_n = 2 + (-1)^n$ **5)** $a_n = \frac{2^n}{2^{n+1}}$ **6)** $a_n = \frac{2^n - 1}{2^n}$

Find a formula for the n^{th} term of the following sequences

1) $1,-1,1,-1,1,....$ **2)** $-1,1,-1,1,-1,....$ **3)** $1,-4,9,-16,25,...$

4) $1,-\frac{1}{4},\frac{1}{9},-\frac{1}{16},\frac{1}{25},...$ **5)** $0,3,8,15,24,...$ **6)** $-3,-2,-1,0,1,...$

7) $1,5,9,13,17,...$ **8)** $2,6,10,14,18,...$ **9)** $1,0,1,0,1,...$

Which of the following sequences converge and which diverge?

1) $a_n = 2 + (0.1)^n$ ***Ans.*** *Converges,* 2

2) $a_n = \dfrac{1-2n}{1+2n}$ ***Ans.*** *Converges,* -1

3) $a_n = \dfrac{1-5n^4}{n^4+8n^3}$ ***Ans.*** *Converges,* -5

4) $a_n = \dfrac{n^2-2n+1}{n-1}$ ***Ans.*** *Diverges*

5) $a_n = 1 + (-1)^n$ ***Ans.*** *Diverges*

Infinite Series

Infinite series are sequences of a special kind: those in which the n^{th}-term is sum of the first n terms of a related sequence.

Example

Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$$

If we denote the above sequence as a_n, and the resultant sequence of the series as then

$$s_1 = a_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2},$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4},$$

as the first three terms of the sequence $\{s_n\}$.

When the sequence $\{s_n\}$ is formed in this way from a given sequence $\{a_n\}$ by rule

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

the result is called an ***Infinite Series.***

- The number $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} ***partial sum*** of the series.
- Instead of $\{s_n\}$, we usually write $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$.
- The series $\sum a_n$ is said to ***converge*** to a number L if and only if

$$L = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

in which case we call L the sum of the series and write

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + \ldots + a_n + \ldots = L$$

If no such limit exists, the series is said to ***diverge***.

Geometric Series

A series of the form

$$a + ar + ar^2 + ar^3 + \ldots + + ar^{n-1} + \ldots$$

is called a ***Geometric Series***. The ratio of any term to the one before it is r. If $|$

the geometric series converges to $a/(1-r)$. If $|r| \geq 1$, the series diverges unless

If $a = 0$, the series converges to 0.

Example

Geometric series with $a = \dfrac{1}{9}$ and $r = \dfrac{1}{3}$.

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right) = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

Geometric series with $a = 4$ and $r = -\dfrac{1}{2}$.

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \ldots = 4\left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots\right)$$

$$= \frac{4}{1+(1/2)} = \frac{8}{3}$$

Example

Determine whether each series converges or diverges. If it converges, find its sum.

(a) $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$, (b) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, (c) $\sum_{n=1}^{\infty}2\left(\cos\frac{\pi}{3}\right)^n$, (d) $\sum_{n=0}^{\infty}\left(\tan\frac{\pi}{4}\right)^n$, (e) $\sum_{n=1}^{\infty}\frac{5(-1)^n}{4^n}$

Solution

(a) Since the series is a geometric series with $r=\frac{2}{3}<1$, so the series is convergent with a sum of $\frac{1}{1-(2/3)}=3$

(b) Since the series is a geometric series with $r=\frac{3}{2}>1$, so the series is divergent.

(c) $\cos\pi/3=1/2$. This is a geometric series with first term $a_1=1$ and the ratio $r=1/2$; so the series converges and its sum is $1/(1-\frac{1}{2})=2$.

(d) $\tan\pi/4=1$. This is a geometric series with $r=1$, so the series diverges.

(e) This is a geometric series with first term $a_1=-5/4$ and ratio $r=-1/4$. So the series converges and its sum is $\frac{-5/4}{1+(1/4)}=-1$.

Test Convergence of Series with Non-negative Terms

1) The n^th^- Term Test

- If $\lim_{n\to\infty}a_n\neq 0$, or if $\lim_{n\to\infty}a_n$ fails to exist, then $\sum_{n=1}^{\infty}a_n$ diverges.
- If $\sum_{n=1}^{\infty}a_n$ converges, then $a_n\to 0$.
- If $\lim_{n\to\infty}a_n=0$, then the test fails.

From the above, it can not be concluded that if $a_n \to 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The series $\sum_{n=1}^{\infty} a_n$ may diverge even though $a_n \to 0$. Thus $\lim_{n\to\infty} a_n = 0$ is a necessary but not a sufficient condition for the series $\sum_{n=1}^{\infty} a_n$ to converge.

Examples

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$,

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1 \neq 0$,

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist,

$\sum_{n=1}^{\infty} \frac{n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+5} = \frac{1}{2} \neq 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ can not be tested by the n$^{\text{th}}$-term test for divergence because $\frac{1}{n} \to 0$.

2) The Ratio Test

Let $\sum a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$$

Then

- The series converges if $\rho<1$,
- The series diverges if $\rho>1$,
- The series may converge or it may diverge if $\rho=1$. (Test fails)

The Ratio Test is often effective when the terms of the series contain factorials of expressions involving n or expressions raised to a power involving n.

Example

Test the following series for convergence or divergence , using the Ratio Test.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n!n!}{(2n)!}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$, (c) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$, (d) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{3^n}$

Solution

(a) If $a_n=\dfrac{n!n!}{(2n)!}$, then $a_{n+1}=\dfrac{(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)(2n+1)(2n)!}=\frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$=\frac{n+1}{4n+2}\to\frac{1}{4}<1 \qquad \textbf{\textit{(Conv.)}}$$

(b) If $a_n = \dfrac{4^n n! n!}{(2n)!}$, then $a_{n+1} = \dfrac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$ and

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \times \frac{(2n)!}{4^n n! n!} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)}$$

$$= \frac{2(n+1)}{2n+1} \to 1$$

(Test fails)

(c) If $a_n = \dfrac{2^n + 5}{3^n}$, then $a_{n+1} = \dfrac{2^{n+1} + 5}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \times \frac{2^{n+1} + 5}{2^n + 5}$$

$$= \frac{1}{3} \times \left(\frac{2 + 5 \times 2^{-n}}{1 + 5 \times 2^{-n}} \right) \to \frac{1}{3} \times \frac{2}{1} = \frac{2}{3} < 1$$

(Conv.)

(d) If $a_n = \dfrac{n!}{3^n}$, then $a_{n+1} = \dfrac{(n+1)!}{3^{n+1}}$ and

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \times \frac{3^n}{n!} = \frac{n+1}{3} \to \infty > 1$$

(Div.)

3) The n^{th} Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n > n_0$ and suppose that

$$\sqrt[n]{a_n} \to \rho$$

Then

- The series converges if $\rho < 1$.
- The series diverges if $\rho > 1$.
- The test is not conclusive if $\rho = 1$.

Example

Test the convergence of the following series using the n[th] Root Test.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$, (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$, (c) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$, (d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (e) $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

Solution

(a) $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \to 0 < 1$ ***(Conv.)***

(b) $\sqrt[n]{\frac{2^n}{n^2}} = \frac{2}{\sqrt[n]{n^2}} = \frac{2}{\left(\sqrt[n]{n}\right)^2} \to \frac{2}{1^2} = 2 > 1$ ***(Div.)***

(c) $\sqrt[n]{\left(1 - \frac{1}{n}\right)^n} = \left(1 - \frac{1}{n}\right) \to 1$ ***(Test fails)***

(d) $\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n \to \frac{1}{e} = \frac{1}{2.7} < 1$ ***(Conv.)***

(e) $\sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = \frac{2n}{n+1} \to 2 > 1$ ***(Div.)***

Exercises on Series

Find the sum of the following series

1) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$ *Ans.* $\frac{4}{5}$

2) $\sum_{n=1}^{\infty} \frac{7}{4^n}$ *Ans.* $\frac{7}{3}$

3) $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n} \right)$ *Ans.* $\frac{23}{2}$

4) $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n} \right)$ *Ans.* $\frac{17}{6}$

Which of the following series converges and which diverges?

$\sum_{n=1}^{\infty} \frac{1}{10^n}$ ***Ans.*** ***Converges (Geometric)***

$\sum_{n=1}^{\infty} \frac{n}{n+1}$ ***Ans.*** ***Diverges (n^{th}-term test)***

$\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ ***Ans.*** ***Converges (Ratio Test)***