Laplace Transform

Let $f(t)$ be a function of t. Then the ***Laplace Transform*** of $f(t)$ is

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Laplace Transform for some Functions

- $f(t) = 1$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \int_0^\infty (1).e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{0}\right] = \frac{1}{s}$$

So, $\mathcal{L}\{k\} = \dfrac{k}{s}$

- $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at}.e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty$$

$$\mathcal{L}\{e^{at}\} = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}$$

- $\cos(at)$, $\sin(at)$

$$e^{jat} = \cos(at) + j\sin(at)$$

$$\mathcal{L}\{e^{jat}\} = \mathcal{L}\{\cos(at)\} + j\mathcal{L}\{\sin(at)\}$$

$$\mathcal{L}\{e^{jat}\} = \frac{1}{s-ja} \times \frac{s+ja}{s+ja} = \frac{s+ja}{s^2+a^2}$$

$$\mathcal{L}\{e^{jat}\} = \frac{s}{s^2+a^2} + j\frac{a}{s^2+a^2}$$

By comparison $\Rightarrow$ $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ & $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

- $f(t) = \sinh(at) = \frac{1}{2}\left(e^{at} - e^{-at}\right)$

$$\mathcal{L}\{\sinh(at)\} = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{1}{2}\frac{s+a-s+a}{s^2-a^2} = \frac{a}{s^2-a^2}$$

- $f(t) = \cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right)$

$$\mathcal{L}\{\cosh(at)\} = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\frac{s+a+s-a}{s^2-a^2} = \frac{s}{s^2-a^2}$$

- $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty t.e^{-st}\,dt$$

$$u = t \quad \Rightarrow \quad du = dt\,, \quad dv = e^{-st}dt \quad \Rightarrow \quad v = -\frac{1}{s}e^{-st}$$

$$\mathcal{L}\{t\} = \frac{-t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\,dt = 0 - 0 - \frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{-1}{s^2}\left(e^{-\infty} - e^0\right)$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

In general, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

- $f(t) = u(t)$

$$\mathcal{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty (1)e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

- $f(t) = u_a(t)$

$$\mathcal{L}\{u_a(t)\} = \int_0^\infty u_a(t)e^{-st}dt = \int_a^\infty (1)e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_a^\infty = \frac{e^{-as}}{s}$$

Laplace Transform Properties

1) Linearity

If $F_1(s) = \mathcal{L}\{f_1(t)\}$ and $F_2(s) = \mathcal{L}\{f_2(t)\}$ then

$$\mathcal{L}\{C_1 f_1(t) + C_2 f_2(t)\} = C_1 F_1(s) + C_2 F_2(s)$$

Example

$$\mathcal{L}\{4t^2 - 3\cos(2t) + 5e^{-t}\} = 4\times\frac{2!}{s^3} - 3\times\frac{s}{s^2+4} + 5\times\frac{1}{s+1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

2) Shifting Property

- If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

- If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{f(t-a)\} = F(s)e^{-as}$$

Example

$$\mathcal{L}\{e^{-t}\cos(2t)\}$$

Here, $f(t) = \cos(2t)$ & $a = -1$, then $F(s) = \dfrac{s}{s^2+4}$ and

$$\mathcal{L}\left\{e^{-t}\cos(2t)\right\} = F(s+1) = \frac{s+1}{(s+1)^2+4}$$

3) Derivative Property

If $F(s) = \mathcal{L}\{f(t)\}$ then

- ❖ $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
- ❖ $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

4) Integral Property

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{\int_0^t f(u)du\right\} = \frac{F(s)}{s}$$

Example

$$\mathcal{L}\left\{\int_0^t \sin(2u)du\right\}$$

Here, $f(t) = \sin(2t)$ then $F(s) = \dfrac{2}{s^2+4}$ and

$$\mathcal{L}\left\{\int_0^t \sin(2u)du\right\} = \frac{F(s)}{s} = \frac{2}{s(s^2+4)}$$

5) Multiplication by t^n

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^nF(s)}{ds^n}$$

Example

$$\mathcal{L}\{t^2 \sin(t)\}$$

Here, $f(t) = \sin(t)$ then $F(s) = \dfrac{1}{s^2+1}$

$$\mathcal{L}\{t^2 \sin(t)\} = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{1}{s^2+1}\right\}$$

$$\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\} = \frac{-2s}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left\{\frac{1}{s^2+1}\right\} = \frac{(s^2+1)^2 \times (-2) + 2s \times 2(s^2+1)(2s)}{(s^2+1)^4} = \frac{(s^2+1)\left[(-2)(s^2+1) + 8s^2\right]}{(s^2+1)^4}$$

$$= \frac{6s^2-2}{(s^2+1)^3}$$

Inverse Laplace Transform

$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$	$F(s)$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1	$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
$\frac{1}{s^2}$	t	$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	$\frac{1}{s^2-a^2}$	$\frac{\sinh(at)}{a}$
$\frac{1}{s-a}$	e^{at}	$\frac{s}{s^2-a^2}$	$\cosh(at)$

Example

Find $f(t)$ if

(a) $F(s)=\dfrac{5}{s+3}$, (b) $F(s)=\dfrac{s+1}{s^2+1}$, (c) $F(s)=\dfrac{1}{(s+25)^2}$,

(d) $F(s)=\dfrac{s+2}{(s+2)^2+1}$, (e) $F(s)=\dfrac{s}{(s-1)^2-4}$, (f) $F(s)=\dfrac{1}{s^2(s^2+1)}$,

(g) $F(s)=\dfrac{4}{s^2+2s+10}$

Solution

(a) $f(t)=5e^{-3t}$

(b) $F(s)=\dfrac{s+1}{s^2+1}=\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1} \quad \Rightarrow \quad f(t)=\cos(t)+\sin(t)$

(c) Using the shifting property $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$ then $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$.

Here, we have $F(s)=\dfrac{1}{s^2}$ with $a=-25$. Since, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}=t$ then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+25)^2}\right\}=t\cdot e^{-25t}$$

(d) Here, we have a shifting of -2 with $F_1(s)=\dfrac{s}{s^2+1}$. So, $f_1(t)=\cos(t)$ and

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}=f(t)=\cos(t)\cdot e^{-2t}$$

(e) $F(s)=\dfrac{s}{(s-1)^2-4}=\dfrac{s-1+1}{(s-1)^2-4}=\dfrac{s-1}{(s-1)^2-4}+\dfrac{1}{(s-1)^2-4}$

So, $f(t)=e^t\cosh(2t)+\dfrac{1}{2}e^t\sinh(2t)$

(f) We know that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}=\sin(t)$ and using the property of division by s which means an integration in time domain, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}=\int_0^t \sin(u)du=1-\cos(t)$$

Again using the same property we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}=\int_0^t (1-\cos(u))du=t-\sin(t)$$

(g) $F(s)=\dfrac{4}{s^2+2s+10}=\dfrac{4}{s^2+2s+1-1+10}=\dfrac{4}{(s+1)^2+9}$

This is $F_1(s)=\dfrac{4}{s^2+9}$ with a shifting of $a=-1$. So, $f_1(t)=\dfrac{4}{3}\sin(3t)$ and $f(t)=\dfrac{4}{3}e^{-t}\sin(3t)$

Solution of Inverse Using Partial Fraction Method

Example

Find $f(t)$ if $F(s)=\dfrac{3s+7}{s^2-2s-3}$

Solution

$$\frac{3s+7}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

$$\Rightarrow \quad 3s+7 = A(s-3) + B(s+1)$$

First Method

$$3s+7 = As - 3A + Bs + B \quad \Rightarrow \quad A + B = 3$$

$$\Rightarrow \quad -3A + B = 7$$

Solving these two equations we get $A = -1$ and $B = 4$

Second Method

$$3s+7 = A(s-3) + B(s+1)$$

At $s = -1$ we get $-3 + 7 = -4A \;\Rightarrow\; A = -1$

At $s = 3$ we get $9 + 7 = 4B \;\Rightarrow\; B = 4$

Third Method

$$A = \left.\frac{3s+7}{(s-3)}\right|_{s=-1} = \frac{3(-1)+7}{-1-3} = \frac{4}{-4} = -1$$

$$B = \left.\frac{3s+7}{(s+1)}\right|_{s=3} = \frac{3(3)+7}{3+1} = \frac{16}{4} = 4$$

$$F(s) = \frac{3s+7}{(s+1)(s-3)} = \frac{-1}{s+1} + \frac{4}{s-3}$$

So $$f(t) = -e^{-t} + 4e^{3t}$$

Example

Find $f(t)$ if $F(s) = \dfrac{3s+1}{(s-1)(s^2+1)}$

Solution

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$A = \left.\frac{3s+1}{s^2+1}\right|_{s=1} = \frac{3(1)+1}{(1)^2+1} = \frac{4}{2} = 2$$

So $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{2}{s-1} + \dfrac{Bs+C}{s^2+1} \Rightarrow 3s+1 = 2(s^2+1) + (Bs+C)(s-1)$

$$3s+1 = 2s^2 + 2 + Bs^2 - Bs + Cs - C$$

$$3s+1 = (2+B)s^2 + (C-B)s + 2 - C \Rightarrow 2+B=0 \Rightarrow B=-2$$

$$\Rightarrow C-B=3 \Rightarrow C=1$$

$$F(s) = \frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$F(s) = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

So $\quad f(t) = 2e^t - 2\cos(t) + \sin(t)$

Note:

If $f(t)$ has the form of $\dfrac{K}{(s-s_i)^n}$ then the partial fraction of it will be

$$\frac{K}{(s-s_i)^n} = \frac{C_1}{(s-s_i)} + \frac{C_2}{(s-s_i)^2} + \ldots (+ \frac{C_{n-1}}{s-s_i)^{n-1}} + \frac{C_n}{(s-s_i)^n}$$

$$C_n = F(s)(s-s_i)^n\Big|_{s=s_i} \qquad C_{n-1} = \frac{1}{1!}\frac{d}{ds}\left[F(s)(s-s_i)^n\right]\Big|_{s=s_i}$$

$$C_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left[F(s)(s-s_i)^n\right]\Big|_{s=s_i}$$

or in general $$C_{n-k} = \frac{1}{k!}\frac{d^k}{ds^k}\left[F(s)(s-s_i)^n\right]\Big|_{s=s_i}$$

Example

Find $f(t)$ if $F(s) = \dfrac{s-1}{(s+1)^3}$

Solution

$$\frac{s-1}{(s+1)^3} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$C = s-1\big|_{s=-1} = -2$$

$$B = \frac{1}{1!}\frac{d}{ds}(s-1)\Big|_{s=-1} = 1$$

$$A = \frac{1}{2!}\frac{d^2}{ds^2}(s-1)\Big|_{s=-1} = \frac{1}{2}\frac{d}{ds}(1)\Big|_{s=-1} = 0$$

So $F(s) = \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3}$

$$\Rightarrow f(t) = e^{-t}\left(t - t^2\right)$$

Example

$$F(s) = \frac{-4s}{(s^2+4)^2} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+4)^2} \qquad (H.W)$$

Another Solution

$$\int \frac{-4s}{(s^2+4)^2}\,ds = -2\int \frac{2s}{(s^2+4)^2}\,ds$$

$$= \frac{-2}{-1}\left(\frac{1}{s^2+4}\right) = \sin(2t)$$

$$\Rightarrow\ f(t) = -t\sin(2t)$$

Example

- $F(s) = \tan^{-1}(s) \qquad \Rightarrow \qquad F'(s) = \dfrac{1}{1+s^2}$

  $t \cdot f(t) \Leftrightarrow -\dfrac{d}{ds}[F(s)] \qquad \Rightarrow \qquad -t \cdot f(t) \Leftrightarrow \dfrac{1}{s^2+1}$

  $-t \cdot f(t) = \sin(t) \qquad \Rightarrow \qquad f(t) = -\dfrac{\sin(t)}{t}$

- $F(s) = \ln(s^2+2) \qquad \Rightarrow \qquad F'(s) = \dfrac{2s}{s^2+2}$

  $-t \cdot f(t) = 2\cos\left(\sqrt{2}t\right) \qquad \Rightarrow \qquad f(t) = \dfrac{-2}{t}\cos\left(\sqrt{2}t\right)$

- $F(s) = \dfrac{e^{-4s}}{s-2}$

  We know that $\mathcal{L}\{g(t-a)u_a(t)\} = G(s)e^{-as}$. Here $a = 4$

  $\dfrac{1}{s-2} \Leftrightarrow e^{2t} = g(t) \qquad \Rightarrow \qquad g(t-4) = e^{2(t-4)}$

  $f(t) = e^{2(t-4)}u_4(t) = e^{2(t-4)}u(t-4)$

Exercises

Find the Inverse Laplace Transform of the following functions

1) $F(s) = \dfrac{1}{s^2+s}$ ***Ans.*** $f(t) = 1 - e^{-t}$

2) $F(s) = \dfrac{1}{s^3+4s}$ ***Ans.*** $f(t) = \left(1+\cos(2t)\right)/4$

3) $F(s) = \dfrac{1}{s}\left(\dfrac{s-a}{s+a}\right)$ ***Ans.*** $f(t) = 2e^{-at} - 1$

4) $F(s) = \dfrac{8}{s^4-4s^2}$ ***Ans.*** $f(t) = \sinh(2t) - 2t$

5) $F(s) = \dfrac{1}{s^4-2s^3}$ ***Ans.*** $f(t) = \left(e^{2t} - 1 - 2t - 2t^2\right)/8$

6) $F(s) = \dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$ ***Ans.*** $f(t) = 1 + t - \cos(t) - \sin(t)$

Solution of Differential Equation Using Laplace Transform

Here, we use the derivative property as follows:

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Example

Solve the following differential equation using Laplace transform

(a) $y'' + 2y' + y = t$ with $y(0) = 0$, and $y'(0) = 1$

(b) $y' + y = 4$ with $y(0) = 0$

(c) $y' + y = \sin(t)$ with $y(0) = 1$

(d) $ty' + y = t$

(e) $ty'' - ty' + y = 0$ with $y(0) = 0$, and $y'(0) = 1$

Solution

(a) Taking the Laplace transform of the two sides, we get

$$\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s^2}$$

$$\left(s^2Y(s) - s(0) - 1\right) + 2\left(sY(s) - 0\right) + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - 1 + 2sY(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1}{s^2} + 1$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1 + s^2}{s^2}$$

$$Y(s)=\frac{1+s^2}{s^2\left(s^2+2s+1\right)}=\frac{1+s^2}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+1)}+\frac{D}{(s+1)^2}$$

$$B=\left.\frac{1+s^2}{(s+1)^2}\right|_{s=0}=1$$

$$A=\frac{d}{ds}\left[\frac{1+s^2}{(s+1)^2}\right]\Bigg|_{s=0}=\left.\frac{(s+1)^2(2s)-2\left(1+s^2\right)(s+1)}{(s+1)^4}\right|_{s=0}=\frac{-2}{1}=-2$$

$$D=\left.\frac{1+s^2}{s^2}\right|_{s=-1}=2$$

$$C=\frac{d}{ds}\left[\frac{1+s^2}{s^2}\right]\Bigg|_{s=-1}=\left.\frac{s^2(2s)-\left(1+s^2\right)(2s)}{s^4}\right|_{s=-1}=\frac{-2+4}{1}=2$$

$$Y(s)=\frac{-2}{s}+\frac{1}{s^2}+\frac{2}{s+1}+\frac{2}{(s+1)^2}\quad\Rightarrow\quad y(t)=-2+t+2e^{-t}+2t\cdot e^{-t}$$

(b) Taking the Laplace transform of the two sides, we get

$$sY(s)-y(0)+Y(s)=\frac{4}{s}\quad\Rightarrow\quad sY(s)-0+Y(s)=\frac{4}{s}$$

$$Y(s)(s+1)=\frac{4}{s}\quad\Rightarrow\quad Y(s)=\frac{4}{s(s+1)}$$

$$\frac{4}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

$$A=\left.\frac{4}{(s+1)}\right|_{s=0}=4,\quad\text{and}\quad B=\left.\frac{4}{s}\right|_{s=-1}=-4$$

$$Y(s)=\frac{4}{s}-\frac{4}{s+1}\quad\Rightarrow\quad y(t)=4-4e^{-t}$$

(c) Taking the Laplace transform of the two sides, we get

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2+1} \quad \Rightarrow \quad Y(s)(s+1) = \frac{1}{s^2+1} + 1$$

$$Y(s)(s+1) = \frac{1+s^2+1}{s^2+1} \quad \Rightarrow \quad Y(s) = \frac{s^2+2}{(s+1)(s^2+1)}$$

$$\frac{s^2+2}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A = \left.\frac{s^2+2}{(s^2+1)}\right|_{s=-1} = \frac{(-1)^2+2}{(-1)^2+1} = \frac{3}{2}$$

$$s^2+2 = A(s^2+1) + (Bs+C)(s+1)$$

$$s^2+2 = (A+B)s^2 + (B+C)s + A + C$$

$$1 = A + B \;\Rightarrow\; B = 1 - \frac{3}{2} = -\frac{1}{2}, \qquad B + C = 0 \;\Rightarrow\; C = \frac{1}{2}$$

$$Y(s) = \frac{3/2}{s+1} + \frac{(-1/2)s + (1/2)}{s^2+1}$$

$$Y(s) = \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} \quad \Rightarrow \quad y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$$

(d) Taking the Laplace transform of the two sides, we get

$$\frac{-d}{ds}\big(sY(s) - y(0)\big) + Y(s) = \frac{1}{s^2} \quad \Rightarrow \quad \frac{-d}{ds}\big(sY(s) - 0\big) + Y(s) = \frac{1}{s^2}$$

$$\frac{-d}{ds}\big(sY(s)\big) + Y(s) = \frac{1}{s^2} \quad \Rightarrow \quad -\big(sY'(s) + Y(s)\big) + Y(s) = \frac{1}{s^2}$$

$$-sY'(s) = \frac{1}{s^2} \quad \Rightarrow \quad Y'(s) = \frac{-1}{s^3} \quad \Rightarrow \quad \frac{dY(s)}{ds} = \frac{-1}{s^3}$$

$$dY(s)=\frac{-1}{s^3}ds \quad\Rightarrow\quad Y(s)=\int\frac{-1}{s^3}ds \;\Rightarrow\quad Y(s)=\frac{1}{2s^2} \quad\Rightarrow\quad y(t)=\frac{1}{2}t$$

(e) Taking the Laplace transform of the two sides, we get

$$\frac{-d}{ds}\left(s^2Y(s)-sy(0)-y'(0)\right)-\frac{-d}{ds}\left(sY(s)-y(0)\right)+Y(s)=0$$

$$\frac{-d}{ds}\left(s^2Y(s)-0-1\right)+\frac{d}{ds}\left(sY(s)-0\right)+Y(s)=0$$

$$\frac{-d}{ds}\left(s^2Y(s)-1\right)+\frac{d}{ds}\left(sY(s)\right)+Y(s)=0$$

$$-\left(s^2Y'(s)+Y(s)\times(2s)\right)+\left(sY'(s)+Y(s)\right)+Y(s)=0$$

$$\left(-s^2+s\right)Y'(s)+\left(-2s+2\right)Y(s)=0 \quad\Rightarrow\quad \left(-s^2+s\right)Y'(s)=\left(2s-2\right)Y(s)$$

$$Y'(s)=\frac{dY(s)}{ds}=\frac{2s-2}{-s^2+s}Y(s) \quad\Rightarrow\quad \frac{dY(s)}{Y(s)}=\frac{2(s-1)}{-s(s-1)}ds$$

$$\int\frac{dY\ s)}{Y(s)}=\int\frac{2}{-s}ds \quad\Rightarrow\quad \ln(Y(s))=-2\ln(s) \quad\Rightarrow\quad \ln(Y(s))=\ln(s^{-2})$$

$$\ln(Y(s))=\ln\left(\frac{1}{s^2}\right) \quad\Rightarrow\quad Y(s)=\frac{1}{s^2} \quad\Rightarrow\quad y(t)=t$$

Example

Solve the following differential equations

(a) $y_1'=-y_2 \qquad y_1(0)=1$

$y_2'=y_1 \qquad y_2(0)=0$

(b) $\dfrac{dx}{dt}=2x-3y \qquad x(0)=8$

$\dfrac{dy}{dt}=y-2x \qquad y(0)=3$

Solution

(a) Taking the Laplace transform of the two equations, we get

$$sY_1(s) - y_1(0) = -Y_2(s) \quad \Rightarrow \quad sY_1(s) - 1 = -Y_2(s) \quad \Rightarrow \quad sY_1(s) + Y_2(s) = 1$$

$$sY_2(s) - y_2(0) = Y_1(s) \quad \Rightarrow \quad sY_2(s) - 0 = Y_1(s) \quad \Rightarrow \quad -Y_1(s) + sY_2(s) = 0$$

$$\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Y_1(s) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{s}{s^2+1} \quad \Rightarrow \quad y_1(t) = \cos(t)$$

$$Y_2(s) = \frac{\begin{vmatrix} s & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{1}{s^2+1} \quad \Rightarrow \quad y_2(t) = \sin(t)$$

(b) Taking the Laplace transform of the two equations, we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$sX(s) - 8 = 2X(s) - 3Y(s)$$

$$(s-2)X(s) + 3Y(s) = 8 \qquad . \quad . \quad . \quad (1)$$

$$sY(s) - y(0) = Y(s) - 2X(s)$$

$$sY(s) - 3 = Y(s) - 2X(s)$$

$$2X(s) + (s-1)Y(s) = 3 \qquad . \quad . \quad . \quad (2)$$

$$\begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$X(s)=\frac{\begin{vmatrix}8 & 3\\ 3 & s-1\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{8(s-1)-3\times 3}{(s-2)(s-1)-3\times 2}=\frac{8s-8-9}{s^2-3s+2-6}=\frac{8s-17}{s^2-3s-4}$$

$$X(s)=\frac{8s-17}{(s+1)(s-4)}=\frac{A}{s+1}+\frac{B}{s-4}$$

$$A=\left.\frac{8s-17}{s-4}\right|_{s=-1}=\frac{8(-1)-17}{-1-4}=\frac{-25}{-5}=5$$

$$B=\left.\frac{8s-17}{s+1}\right|_{s=4}=\frac{8(4)-17}{4+1}=\frac{15}{5}=3$$

$$X(s)=\frac{5}{s+1}+\frac{3}{s-4}\quad\Rightarrow\quad x(t)=5e^{-t}+3e^{4t}$$

$$Y(s)=\frac{\begin{vmatrix}s-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{3(s-2)-8\times 2}{(s-2)(s-1)-3\times 2}=\frac{3s-6-16}{s^2-3s+2-6}=\frac{3s-22}{s^2-3s-4}$$

$$Y(s)=\frac{3s-22}{(s+1)(s-4)}=\frac{C}{s+1}+\frac{D}{s-4}$$

$$C=\left.\frac{3s-22}{s-4}\right|_{s=-1}=\frac{3(-1)-22}{-1-4}=\frac{-25}{-5}=5$$

$$D=\left.\frac{3s-22}{s+1}\right|_{s=4}=\frac{3(4)-22}{4+1}=\frac{-10}{5}=-2$$

$$Y(s)=\frac{5}{s+1}-\frac{2}{s-4}\quad\Rightarrow\quad y(t)=5e^{-t}-2e^{4t}$$

Exercises

Find the solution of the following Differential Equations

1) $4y'' + \pi^2 y = 0$,	$y(0) = 2$,	$y'(0) = 0$.
2) $y'' + \omega^2 y = 0$,	$y(0) = A$,	$y'(0) = B$.
3) $y'' + 2y' - 8y = 0$,	$y(0) = 1$,	$y'(0) = 8$.
4) $y'' - 2y' - 3y = 0$,	$y(0) = 1$,	$y'(0) = 7$.
5) $y'' - ky' = 0$,	$y(0) = 2$,	$y'(0) = k$.
6) $y'' + ky' - 2k^2 y = 0$,	$y(0) = 2$,	$y'(0) = 2k$.
7) $y' + 4y = 0$,	$y(0) = 2.8$	
8) $y' + \frac{1}{2} y = 17\sin(2t)$,	$y(0) = -1$.	
9) $y'' - y' - 6y = 0$,	$y(0) = 6$,	$y'(0) = 13$.
10) $y'' - \frac{1}{4} y = 0$,	$y(0) = 4$,	$y'(0) = 0$.
11) $y'' - 4y' + 4y = 0$,	$y(0) = 2.1$,	$y'(0) = 3.9$
12) $y'' + 2y' + 2y = 0$,	$y(0) = 1$,	$y'(0) = -3$.
13) $y'' + 7y' + 12y = 21e^{3t}$,	$y(0) = 3.5$,	$y'(0) = -10$.
14) $y'' + 9y = 10e^{-t}$,	$y(0) = 0$,	$y'(0) = 0$.
15) $y'' + 3y' + 2.25y = 9t^3 + 64$,	$y(0) = 1$,	$y'(0) = 31.5$
16) $y'' - 6y' + 5y = 29\cos(2t)$,	$y(0) = 3.2$,	$y'(0) = 6.2$
17) $y'' + 2y' + 2y = 0$,	$y(0) = 0$,	$y'(0) = 1$.
18) $y'' + 2y' + 17y = 0$,	$y(0) = 0$,	$y'(0) = 12$.