# Computing Taylor Series

Lecture Notes

As we have seen, many different functions can be expressed as power series. However, we do not yet have an explanation for some of our series (e.g. the series for $e^x$, $\sin x$, and $\cos x$), and we do not have a general formula for finding Taylor series. In this section we will learn how to find a Taylor series for virtually any function.

## The Taylor Series Formula

A general power series can be expressed as

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots,$$ Centered at $x=0$

where $c_0, c_1, c_2, \ldots$ are constants. As with a polynomial, we often don't bother to write terms that have a coefficient of 0, but we can imagine that every power series has every one of these terms.

The first term of a power series is called the ***constant term***. The constant term is what you get when you substitute in $x = 0$. For example, if

$$f(x) = 3 + 5x + 7x^2 + 9x^3 + 11x^4 + \cdots,$$

then the constant term of $f(x)$ is 3, so

$$f(0) = 3 + 0 + 0 + 0 + 0 + \cdots = 3.$$

The second term of a power series is called the ***linear term*** or ***$x$ term***, and has the form $c_1 x$ for some coefficient $c_1$. You can obtain the coefficient $c_1$ by taking the *derivative* of the series and then substituting $x = 0$. For instance, if

$$f(x) = 3 + 5x + 7x^2 + 9x^3 + 11x^4 + \cdots$$

then

$$f'(x) = 5 + 14x + 27x^2 + 44x^3 + \cdots,$$

so $f'(0) = 5$. As you can see, the coefficient of $x$ in $f(x)$ is the same as the constant term of $f'(x)$, and is therefore equal to $f'(0)$.

In general, taking the derivative of a power series "demotes" each of the coefficients by one step:

General Case

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

$$
\begin{array}{ccccccccccc}
f(x) & = & 3 & + & 5x & + & 7x^2 & + & 9x^3 & + \cdots \\
 & & & & \downarrow & & \downarrow & & \downarrow & \cdots \\
f'(x) & = & & & 5 & + & 14x & + & 27x^2 & + \cdots
\end{array}
$$

The coefficient of $x$ becomes the constant term, the coefficient of $x^2$ becomes the coefficient of $x$ (and is multiplied by 2), the coefficient of $x^3$ becomes the coefficient of $x^2$ (and is multiplied by 3), and so forth.

The following formula relates the coefficients of a power series to the values of the derivatives at 0:

FORMULA FOR THE COEFFICIENTS

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series. Then:

$$c_n = \frac{f^{(n)}(0)}{n!}$$

where $f^{(n)}$ denotes the $n$th derivative of $f$.

The following calculation illustrates this pattern:

$$
\begin{aligned}
\text{If } f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots, \\
\text{then } f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots, \\
f''(x) &= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots, \\
f^{(3)}(x) &= 6c_3 + 24c_4 x + 60c_5 x^2 + \cdots, \\
f^{(4)}(x) &= 24c_4 + 120c_5 x + \cdots, \\
f^{(5)}(x) &= 120c_5 + \cdots, \\
&\vdots
\end{aligned}
$$

As you can see, the constant term of $f^{(n)}(x)$ is always equal to $n!$ multiplied by $c_n$:

$$f^{(n)}(0) = n!\, c_n$$

This explains the formula for the coefficients given above.

The formula above can be used to find a Taylor series for virtually any function. In general, a function is called **analytic** if it can somehow be represented by a power series. Most functions defined by a formula are analytic, and we now know how to find the Taylor series for any analytic function:

TAYLOR SERIES FORMULA

Let $f(x)$ be any function, and suppose that $f(x)$ is analytic. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

EXAMPLE 1 Assuming that $e^x$ is analytic, find the Taylor series for $e^x$.

SOLUTION Let $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, and so on, so

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \ldots$$

We conclude that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \qquad \blacksquare$$

EXAMPLE 2 Assuming that $\sin x$ is analytic, find the Taylor series for $\sin x$.

SOLUTION Let $f(x) = \sin x$. Here are the first seven derivatives:

$$\begin{array}{rclcrcrcl}
f(x) &=& \sin x & \text{so} & f(0) &=& \sin 0 &=& 0 \\
f'(x) &=& \cos x & \text{so} & f'(0) &=& \cos 0 &=& 1 \\
f''(x) &=& -\sin x & \text{so} & f''(0) &=& -\sin 0 &=& 0 \\
f^{(3)}(x) &=& -\cos x & \text{so} & f^{(3)}(0) &=& -\cos 0 &=& -1 \\
f^{(4)}(x) &=& \sin x & \text{so} & f^{(4)}(0) &=& \sin 0 &=& 0 \\
f^{(5)}(x) &=& \cos x & \text{so} & f^{(5)}(0) &=& \cos 0 &=& 1 \\
f^{(6)}(x) &=& -\sin x & \text{so} & f^{(6)}(0) &=& -\sin 0 &=& 0 \\
f^{(7)}(x) &=& -\cos x & \text{so} & f^{(7)}(0) &=& -\cos 0 &=& -1
\end{array}$$

This pattern will continue to repeat. Therefore, the Taylor series for $\sin x$ is:

$$\sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$$

$n = 2$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$$

$$\frac{(-1)^3 x^5}{5!}$$

Don't forget that there are other ways to find the Taylor series for a function. You only need to use the formula if no other ~~method is available.~~

EXAMPLE 3 Find the Taylor series for $\tan^{-1}(x^2)$.

SOLUTION There is no need to use the Taylor series formula here. We can obtain a power series for $\tan^{-1}(x^2)$ by plugging $x^2$ into the Taylor series for $\tan^{-1}(x)$:

بديل

$$\tan^{-1}(x^2) = x^2 - \frac{1}{3}x^6 + \frac{1}{5}x^{10} - \frac{1}{7}x^{14} + \cdots \qquad \blacksquare$$

EXAMPLE 4 Find the Taylor series for $f(x) = \dfrac{1}{(1+x)^2}$.

SOLUTION:

$$\begin{aligned}
f(x) &= \frac{1}{(1+x)^2} &\quad\text{so}\quad f(0) &= 1\\
f'(x) &= -2(1+x)^{-3} &\quad\text{so}\quad f'(0) &= -2\\
f''(x) &= 6(1+x)^{-4} &\quad\text{so}\quad f''(0) &= 6\\
f^{(3)}(x) &= -24(1+x)^{-5} &\quad\text{so}\quad f^{(3)}(0) &= -24\\
f^{(4)}(x) &= 120(1+x)^{-6} &\quad\text{so}\quad f^{(4)}(0) &= 120
\end{aligned}$$

Therefore,

$$\frac{1}{(1+x)^2} = 1 + -2x + \frac{6}{2!}x^2 + \frac{-24}{3!}x^3 + \frac{120}{4!}x^4 + \cdots = 1 + \sum_{n=1}^{\infty}(n+1)(-1)^n x^n$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \qquad \blacksquare$$

In the following example, it is somewhat complicated to find a pattern in the coefficients, making it difficult to find more than the first few terms:

EXAMPLE 5 Find the first three terms of the Taylor series for $f(x) = \sqrt{1+x}$.

SOLUTION

$$\begin{aligned}
f(x) &= (1+x)^{1/2} &\quad\text{so}\quad f(0) &= 1\\
f'(x) &= \frac{1}{2}(1+x)^{-1/2} &\quad\text{so}\quad f'(0) &= \frac{1}{2}\\
f''(x) &= -\frac{1}{4}(1+x)^{-3/2} &\quad\text{so}\quad f''(0) &= -\frac{1}{4}
\end{aligned}$$

$$\frac{(-1)^{n+1}(n+1)x^{n+2}}{n!}$$

$$f(x) = -x^2 + 2x^3 + \frac{3x^4}{2!} + \frac{4x^5}{3!} - \frac{5x^6}{4!} + \cdots$$

Therefore, the first three terms of the Taylor series for $\sqrt{1+x}$ are:

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

Be aware that many functions still cannot be expressed as power series using this formula. For example, the function $f(x) = 1/x$ has no Taylor series, since $f(0)$ is undefined. In general, any function for which $f^{(n)}(0)$ is undefined for some $n$ will fail to be analytic.

## General Taylor Series

So far, we have only been dealing with power series centered at $x = 0$:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

Such a series tends to converge when $x$ is close to 0, and diverge when $x$ is far away from 0.

A more general form for a power series is:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

This is called a power series ***centered at $x = a$***. The advantage of this series is that it tends to converge when $x$ is close to $a$.

For a power series centered at $x = a$, the formula for the $n$th coefficient is

$$c_n = \frac{f^{(n)}(a)}{n!}$$

GENERAL TAYLOR SERIES

Let $f(x)$ be a function, and suppose that $f$ is analytic at $x = a$. Then:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \overset{a \neq 0}{} = \sum_{n=0}^{\infty} C_n x^n \quad a=0$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

The formula above uses the phrase "analytic at $x = a$", which means that $f$ can be expressed as a power series centered at $x = a$.

$f(x) = x^{-2}$

$f'(x) = -2x^{3}$

EXAMPLE 6 Find the Taylor series for $f(x) = 1/x^2$ centered at $x = 1$.

SOLUTION We have:

$$\begin{aligned} f(x) &= 1/x^2 & \text{so} \quad f(1) &= 1 \\ f'(x) &= -2x^{-3} & \text{so} \quad f'(1) &= -2 \\ f''(x) &= 6x^{-4} & \text{so} \quad f''(1) &= 6 \\ f^{(3)}(x) &= -24x^{-5} & \text{so} \quad f^{(3)}(1) &= -24 \\ f^{(4)}(x) &= 120x^{-6} & \text{so} \quad f^{(4)}(1) &= 120 \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{x^2} &= 1 + -2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{-24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 + \cdots \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots \quad \blacksquare \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n (n+1)(x-1)^n \; ! \end{aligned}$$

It is also possible to obtain a Taylor series centered at $x = a$ using substitution. For example, we know the formula

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$

Plugging in $x - 1$ for $x$ gives the Taylor series for $\ln x$ centered at $x = 1$:

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

Note that $\ln x$ does not have a Taylor series centered at $x = 0$, since $\ln(0)$ is undefined.

Absolute convergence, Conditional convergance, Divergance

Does: $\sum a_n$ converge or Diverage ?

Results: 1) Suppose $\sum |a_n|$ converges; if so the original series $\sum a_n$ also converges, and call $\sum a_n$ absolutely convergent.

2) Suppose $\sum |a_n|$ diverges, but $\sum a_n$ converges; if so, we call $\sum a_n$ conditionally convergent.

3) If both $\sum |a_n|$ and $\sum a_n$ diverge, we call $\sum a_n$ divergent.

$\therefore \ln x = \frac{1}{n}(-1)^{n+1}(x-1)^n$

## EXERCISES

1. Let $f(x) = e^{3x}$.

   (a) Find $f(0)$, $f'(0)$, $f''(0)$, and $f^{(3)}(0)$.

   (b) What is the general formula for $f^{(n)}(0)$?

   (c) Use your answer from part (b) to find the Taylor series for $e^{3x}$.

2. Let $f(x) = \dfrac{1}{1+x}$.

   (a) Find $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, $f^{(4)}(0)$, and $f^{(5)}(0)$.

   (b) What is the general formula for $f^{(n)}(0)$?

   (c) Use your answer from part (b) to find the Taylor series for $\dfrac{1}{1+x}$.

3. Find the Taylor series for $f(x) = \dfrac{2}{(1+x)^3}$. Express your answer using summation notation.

4. Find the Taylor series for $f(x) = \dfrac{6}{(1-x)^4}$. Express your answer using summation notation.

5. Find the first four terms of the Taylor series for $\sqrt[3]{1+x}$.

6. Find the first four terms of the Taylor series for $\sqrt{x+4}$.

7. Use the Taylor series formula to find the Taylor series for $\cos x$.

8. Use the Taylor series formula to find the Taylor series for $\ln(1+x)$.

**9–14** ▪ Find the Taylor series for $f(x)$ without using the Taylor series formula. Express your answer using summation notation.

AS

9. $f(x) = e^{5x}$

10. $f(x) = 2^x$

11. $f(x) = \ln(x+e)$

12. $f(x) = \sin^2(x)$

13. $f(x) = x^3\sin(2x)$

14. $f(x) = \int_0^x e^{-t^2}\,dt$

**15–18** ▪ Find the first three terms of the Taylor series for $f(x)$ centered at the given value of $a$.

15. $f(x) = \sqrt{x}$, $a = 25$

16. $f(x) = \sqrt[3]{x}$, $a = 8$

17. $f(x) = \tan^{-1} x$, $a = 1$

18. $f(x) = \tan x$, $a = \dfrac{\pi}{4}$

**19–26** ▪ Find the Taylor series for $f(x)$ centered at the given value of $a$.

19. $f(x) = e^x$, $a = 3$

20. $f(x) = e^{2x}$, $a = 5$

21. $f(x) = \sin x$, $a = -\dfrac{\pi}{2}$ ✓

22. $f(x) = \cos x$, $a = \dfrac{\pi}{2}$ ✓

23. $f(x) = x^4$, $a = 2$

24. $f(x) = (x-5)^3$, $a = 5$

25. $f(x) = \dfrac{1}{x}$, $a = 1$

26. $f(x) = \dfrac{1}{x}$, $a = -7$

$2x^2 = 4x^3$

$x^2 = 2x$

$= x^3$

$3x^2$

$f(x) = (x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x)^{-\frac{1}{2}}$ $\quad -\frac{3}{2}$ $\quad 1.5$

$f''(x) = -\frac{1}{4}(x)^{-2.5}$

$f'''(x) = \frac{3}{8}(x)$

Ex $\dfrac{n^n}{3(1+3n)}$

$\dfrac{(n+1)^{n+1}}{3[1+3n+3]} * \dfrac{3(1+3n)}{n^n}$

$\dfrac{n^{n+1} * 1^{n+1}}{n^n} * \dfrac{9n+3}{9n+12} = \infty$

# Applications of Taylor Series

Lecture Notes

These notes discuss three important applications of Taylor series:

1. Using Taylor series to find the sum of a series.
2. Using Taylor series to evaluate limits.
3. Using Taylor polynomials to approximate functions.

## Evaluating Infinite Series

It is possible to use Taylor series to find the sums of many different infinite series. The following examples illustrate this idea.

EXAMPLE 1 Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

SOLUTION Recall the Taylor series for $e^x$:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x.$$

The sum of the given series can be obtained by substituting in $x = 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e.$$

■

In the above example, note that we get a different series for every value of $x$ that we plug in. For example,

$$1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots = e^2.$$

and

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}.$$

EXAMPLE 2 Find the sums of the following series:

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$. (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

SOLUTION

(a) Recall that

$$x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} - \cdots = \ln(1 + x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2).$$

(b) Recall that

$$x - \frac{x}{3} + \frac{x}{5} - \frac{x}{7} + \frac{x}{9} - \cdots = \tan^{-1}(x).$$

Substituting in $x = 1$ yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

This is known as the **Gregory-Leibniz formula** for $\pi$. ■

## Limits Using Power Series

When taking a limit as $x \to 0$, you can often simplify things by substituting in a power series that you know.

EXAMPLE 3 Evaluate $\lim\limits_{x \to 0} \frac{\sin x - x}{x^3}$.

SOLUTION We simply plug in the Taylor series for $\sin x$:

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^3}$$

$$= \lim_{x \to 0} -\frac{1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + \cdots = -\frac{1}{3!} = -\frac{1}{6}$$ ■

Plugging this in gives

$$\begin{aligned}
\lim_{x\to 1}\frac{\ln x}{x-1} &= \lim_{x\to 1}\frac{(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\cdots}{x-1} \\
&= \lim_{x\to 1}\left(1-\frac{1}{2}(x-1)+\frac{1}{3}(x-1)^2+\cdots\right) = 1 \qquad \blacksquare
\end{aligned}$$

## Taylor Polynomials

A partial sum of a Taylor series is called a **Taylor polynomial**. For example, the Taylor polynomials for $e^x$ are:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x + \frac{1}{2}x^2 \\
T_3(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
&\vdots
\end{aligned}$$

You can approximate any function $f(x)$ by its Taylor polynomial:

$$f(x) \approx T_n(x)$$

If you use the Taylor polynomial centered at $a$, then the approximation will be particularly good near $x = a$.

### TAYLOR POLYNOMIALS

Let $f(x)$ be a function. The **Taylor polynomials** for $f(x)$ centered at $x = a$ are:

$$\begin{aligned}
T_0(x) &= f(a) \\
T_1(x) &= f(a) + f'(a)(x-a) \\
T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\
&\vdots
\end{aligned}$$

You can approximate $f(x)$ using a Taylor polynomial.

Note that the 1st-degree Taylor polynomial is just the tangent line to $f(x)$ at $x = a$:

$$T_1(x) = f(a) + f'(a)(x-a)$$

This is often called the **linear approximation** to $f(x)$ near $x = a$, i.e. the tangent line to the graph. Taylor polynomials can be viewed as a generalization of linear approximations. In particular, the 2nd-degree Taylor polynomial is sometimes called the **quadratic approximation**, the 3rd-degree Taylor polynomial is the **cubic approximation**, and so on.

EXAMPLE 7

(a) Find the 5th-degree Taylor polynomial for $\sin x$.

(b) Use the answer from part (a) to approximate $\sin(0.3)$.

SOLUTION

(a) This is just all term terms of the Taylor series up to $x^5$:

$$T_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

(b) $\sin(0.3) \approx T_5(0.3) = (0.3) - \frac{1}{6}(0.3)^3 + \frac{1}{120}(0.3)^5 = 0.295\,520\,25$ ■

## XERCISES

2 ■ Find the sum of the given series.

1. $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$

2. $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$

–12 ■ Evaluate the following limits.

3. $\lim\limits_{x\to 0} \frac{\cos x - 1}{x^2}$

4. $\lim\limits_{x\to 0} \frac{x}{e^{3x} - 1}$

5. $\lim\limits_{x\to 0} \frac{\ln(1+x^2)}{x^2}$

6. $\lim\limits_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

7. $\lim\limits_{x\to 0} \frac{\sin(4x)}{x}$

8. $\lim\limits_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

9. $\lim\limits_{x\to 0} \frac{e^{2x} - 1}{\sin x}$

10. $\lim\limits_{x\to 0} \frac{\tan^{-1}(x) - x}{\sin(x) - x}$

11. $\lim\limits_{x\to \pi} \frac{1 + \cos x}{(x-\pi)^2}$

12. $\lim\limits_{x\to 1} \frac{\ln x}{\sqrt{x} - 1}$

13. (a) Find the 3rd-degree Taylor polynomial for th
$f(x) = \ln x$ centered at $a = 1$.
(b) Use your answer from part (a) to approximate

14. (a) Find the 4th-degree Taylor polynomial for $e^{-}$
(b) Use your answer from part (a) to approximat

15. (a) Find the quadratic approximation for th
$f(x) = x^{3/2}$ centered at $a = 4$.
(b) Use your answer from part (a) to approxima

16. (a) Find the quadratic approximation for t
$f(x) = \sqrt[3]{x}$ centered at $a = 8$.
(b) Use your answer from part (a) to approxim