# Differential Equation

A differential equation is an equation that involves one or more derivatives, or differentials. Differential equations are classified by:

1. ***Type:*** Ordinary or partial.
2. ***Order:*** The order of differential equation is the highest order derivative that occurs in the equation.
3. ***Degree:*** The exponent of the highest power of the highest order derivative.

A differential equation is an **ordinary D.Eqs**. if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variable, the D.Eqs. is a **partial D.Eqs**..

**Ex1:**

$\frac{dy}{dx} = 5x + 3$ 1st order-1st degree

**Ex2:**

$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx}\right)^5$ 3rd order-2nd degree

**Ex3:**

$4\frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} + 5xy = 0$ 3rd order-1st degree

Ex.(4) **:** Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + \cos x = 0$ ans:1st order-1st degree
2. $3dx + 4y^2 dy = 0$ ans:1st order-1st degree
3. $\frac{d^2y}{dx^2} + y = y^2$
4. $(y'')^2 + 2y' = x^2$
5. $y''' + 2(y'')^2 = xy$

**6)** $\frac{dy}{dx} = 5y$

**7)** $3\frac{dy}{dx} - \sin x = 0$

**8)** $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 - \frac{dy}{dx} = e^x$

H.w (1)

1. Show that $y=3e^{2x}-e^{-2x}$ is a solution to $y''-4y=0$
2. Determine whether $y(x)= 2e^{-x}+xe^{-x}$ is a solution of $y''+2y'+y=0$
3. Determine whether $y= x^2-1$ is a solution of $(y')^4+y^2=-1$

**4. Check if $y = e^{-x}$ is a solution for the D.E. $y'' + 3y' + 2y = 0$**

## *Ordinary Differential Equation:*

*Ordinary Differential Equations are Equations involve derivatives and solved by four methods:*

- *Variable Separable.*
- *Homogenous.*
- *Linear.*
- *Exact.*

### 1- Variable Separable:

A first order D.Eq. can be solved by integration if it is possible to collect all y terms with dy and all x terms with dx, that is, if it is possible to write the D.Eq. in the form

$$f(x)dx + g(y)dy = 0$$

then the general solution is:

$$\int f(x)dx + \int g(y)dy = c$$ where **c** is an arbitrary constant.

**Ex.1:**

Solve $\frac{dy}{dx} = e^{x+y}$

**Sol.:**

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\frac{dy}{e^y} = e^x dx$$

$$\int e^{-y}dy = \int e^x \cdot dx$$

$$-\int e^{-y} \cdot (-dy) = \int e^x dx \quad \Rightarrow \quad -e^{-y} = e^x + c$$

H.w (2): Solve The following D.E. using the separating of variables method

1. $x(2y-3)dx+(x^2+1)dy=0$ ans: $(x^2+1)(2y-3)=c$
2. $dy=e^{x-y}\,dx$ ans: $e^y=e^x+c$
3. $\sin x + \cosh 2y\,\frac{dy}{dx} = 0$ ans: $\sinh 2y-2\cos x=c$
4. $xe^y dy+\frac{x^2+1}{y}dx = 0$ ans: $e^y(y-1)+\frac{x^2}{2}+\ln|x|=c$
5. $\sqrt{2xy}\frac{dy}{dx} = 1$ ans: $\frac{\sqrt{2}}{3}y^{\frac{3}{2}} = x^{\frac{1}{2}} + c$

## ***Homogeneous Function*** **(λ)**

If $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ then $f(x, y)$ is homogeneous function and $n$ represents the degree of the homogeneous function.

***Example***

For the function $f(x, y) = x^2 + y^2$ then

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2$$
$$= \lambda^2 x^2 + \lambda^2 y^2$$
$$= \lambda^2 \left(x^2 + y^2\right) = \lambda^2 f(x, y)$$

So, the function $f(x, y)$ is homogeneous with degree $2$.

***Example***

For the function $f(x, y) = x + y^2$ then

$$f(\lambda x, \lambda y) = \lambda x + (\lambda y)^2$$
$$= \lambda x + \lambda^2 y^2$$
$$= \lambda\left(x + \lambda y^2\right)$$

So, the function $f(x, y)$ is not homogeneous.

## 2- Homogeneous Equation:

Some times a D.Eq. which variables can't be separated can be transformed by a change of variables into an equation which variables can be separated. This is the case with any equation that can be put into form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \ldots(1)$$

Such an equation is called homogenous.

Put $\frac{y}{x} = u \Rightarrow$ y $= ux$, $\frac{dy}{dx} = u + x\cdot\frac{du}{dx}$ and (1) becomes

$$x\cdot\frac{du}{dx} + u = f(u)$$

**Ex.1:**

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

**Sol.:**

$$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{y}{x}} \Rightarrow \text{homo. Put } \frac{y}{x} = u \Rightarrow \frac{dy}{dx} = x\cdot\frac{du}{dx} + u$$

$$x\cdot\frac{du}{dx} + u = \frac{1+u^2}{u} \Rightarrow x\cdot\frac{du}{dx} = \frac{1+u^2-u^2}{u}$$

$$x\cdot\frac{du}{dx} = \frac{1}{u} \quad , \quad \int u\cdot du = \int\frac{dx}{x}$$

$$\frac{u^2}{2} = \ln x + c \Rightarrow \frac{y^2}{2x^2} = \ln x + c$$

Ex.(2): Show that the following differential equations are homogenous and solve.

1. $(x^2+y^2)dx+xy\,dy=0$ ans: $x^2(x^2+2y^2)=c$
2. $x^2dy+(y^2-xy)dx=0$ ans: $y = \frac{x}{\ln x - c}$
3. $(xe^{\frac{y}{x}} + y)dx - xdy = 0$ ans: $\ln|x| + e^{\frac{-y}{x}} = c$