Partial Derivatives

Functions of Independent Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, ..., x_n)$. A ***real-valued function*** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, ..., x_n)$$

to each element in D. The set D is the function's ***domain***. The set of w-values taken on by f is the function's ***range***. The symbol w is the ***dependent variable*** of f, and f is said to be a function of the n ***independent variables*** x_1 to x_n. We also call the x_j's the function's ***input variables*** and call w the function's ***output variable***.

Example

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3,0,4)$ is

$$f(3,0,4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5$$

Domains and Ranges

Example

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Both x & y $(-\infty, +\infty)$ or Entire plane	$[-1, +1]$

$Z = f(x, y)$ or $f(x,y,z) = 0$

$$\left.\begin{aligned} \frac{\partial Z}{\partial x} &= Z_x = f_x \\ \frac{\partial Z}{\partial y} &= Z_y = f_y \end{aligned}\right\} \quad 1^{st} \text{ partial derivatives}$$

$$\left.\begin{aligned} \frac{\partial^2 Z}{\partial x^2} &= Z_{xx} = f_{xx} \\ \frac{\partial^2 Z}{\partial y^2} &= Z_{yy} = f_{yy} \\ \frac{\partial^2 Z}{\partial y \partial x} &= Z_{yx} \\ \frac{\partial^2 Z}{\partial x \partial y} &= Z_{xy} \end{aligned}\right\} \quad 2^{nd} \text{ partial derivatives}$$

$$Z_{xy} = Z_{yx}$$

A) ***First Order Partial Derivatives:***

Example

Find $\partial f / \partial y$ if $f(x, y) = y \sin(xy)$

Solution

We treat x as a constant and f as a product of y and $\sin(xy)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y \sin(xy)\right) = y \frac{\partial}{\partial y} \sin(xy) + \sin(xy) \frac{\partial}{\partial y}(y)$$

$$= \left(y \cos(xy)\right) \frac{\partial}{\partial y}(xy) + \sin(xy) = xy \cos(xy) + \sin(xy)$$

Example

Find f_x and f_y if $f(x,y)=\dfrac{2y}{y+\cos x}$

Solution

We treat f as a quotient

$$f_x=\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\dfrac{\partial}{\partial x}(2y)-2y\dfrac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}=\frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y=\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\dfrac{\partial}{\partial y}(2y)-2y\dfrac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}=\frac{2\cos x}{(y+\cos x)^2}$$

Ex.1

If $Z=x^y$, find $\dfrac{\partial Z}{\partial x}$, $\dfrac{\partial Z}{\partial y}$

$\dfrac{\partial Z}{\partial x}=\text{y}\,x^{y\text{-}1}$ y constant , $\dfrac{\partial Z}{\partial y}=x^y\cdot\ln x\cdot dy$, x constant $\Rightarrow$ powerfunction

B)Second Order Partial Derivatives:

Ex.2

If $Z = \tan^{-1}\frac{y}{x}$, show that $Z_{yx} = Z_{xy}$

$$Z_x = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{-\mathbf{y}}{\mathbf{x^2+y^2}}$$

$$Z_{yx} = \frac{(x^2+y^2)(-1)+y.2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \cdots(1)$$

$$Z_y = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{\mathbf{x}}{\mathbf{x^2+y^2}}$$

$$Z_{xy} = \frac{(x^2+y^2)(1)+x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \cdots(2)$$

(1) & (2) are equal

The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout a region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example

If $f(x, y) = x\cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y},$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \cos y + ye^x, \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = -x\sin y + e^x, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

Partial Derivatives of Higher Order

Example

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$

Solution

We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z

$$f_y = -4xyz + x^2, \qquad f_{yx} = -4yz + 2x, \qquad f_{yxy} = -4z, \qquad f_{yxyz} = -4$$

<u>***Homework***</u>

Find the Derivatives of functions below:

1) $f(x,y)=\dfrac{1}{x+y}$ **Ans.** $f_x=\dfrac{-1}{(x+y)^2}, f_y=\dfrac{-1}{(x+y)^2}$

2) $f(x,y)=\dfrac{x+y}{xy-1}$ **Ans.** $f_x=\dfrac{-y^2-1}{(xy-1)^2}, f_y=\dfrac{-x^2-1}{(xy-1)^2}$

3) $f(x,y)=e^{(x+y+1)}$ **Ans.** $f_x=e^{(x+y+1)},\ f_y=e^{(x+y+1)}$

4) $f(x,y)=\ln(x+y)$ **Ans.** $f_x=\dfrac{1}{x+y},\ f_y=\dfrac{1}{x+y}$

5) $f(x,y)=2x^2-3y-4$ **Ans.** $f_x=4x,\ f_y=-3$

6) $f(x,y)=(x^2-1)(y+2)$ **Ans.** $f_x=2x(y+2),\ f_y=x^2-1$

7) $f(x,y)=(xy-1)^2$ **Ans.** $f_x=2y(xy-1), f_y=2x(xy-1)$

8) $f(x,y)=\sqrt{x^2+y^2}$ **Ans.** $f_x=\dfrac{x}{\sqrt{x^2+y^2}}, f_y=\dfrac{y}{\sqrt{x^2+y^2}}$

9) $f(x,y)=x+y+xy$ **Ans.** $f_{xx}=0, f_{yy}=0, f_{xy}=1$

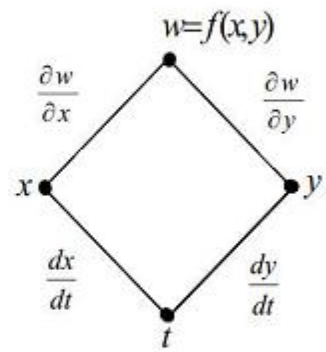
10) $g(x,y)=x^2y+\cos(y)+y\sin(x)$ **Ans.** $g_{xx}=2y-y\sin(x)$,

$g_{yy}=-\cos(y)$,

$g_{xy}=2x+\cos(x)$

Chain Rule For Functions of (Two Or Three)Independent Variables:

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example

Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path

$$x = \cos(t) \quad \& \quad y = \sin(t)$$

What is the derivative's value at $t = \pi/2$?

Solution

We apply the Chain Rule to find dw/dt as follows

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(xy) \times \frac{d}{dt}(\cos(t)) + \frac{\partial}{\partial y}(xy) \times \frac{d}{dt}(\sin(t))$$

$$= y \times (-\sin(t)) + x \times (\cos(t))$$

$$= (\sin(t)) \times (-\sin(t)) + (\cos(t)) \times (\cos(t))$$

$$= -\sin^2(t) + \cos^2(t)$$

$$= \cos(2t)$$

We can check the result with a more direct calculation as a function of t

$$w = xy = \cos(t).\sin(t) = \frac{1}{2}\sin(2t)$$

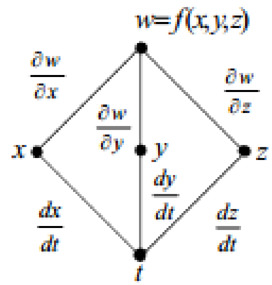
So, $$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin(2t)\right) = \frac{1}{2} \times 2\cos(2t) = \cos(2t)$$

In either case, at a given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2 \times \frac{\pi}{2}\right) = \cos\pi = -1$$

When The Functions have Three independent Variables as shown Below:

There are three routes from w to t instead of two, but finding dw/dt is still the same. Read each route, multiplying derivatives along the way; then add.



$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example

Find dw/dt if

$w = xy + z$, $\quad x = \cos(t)$, $\quad y = \sin(t)$, $\quad z = t$

What is the derivative's value at $t = 0$?

Solution

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

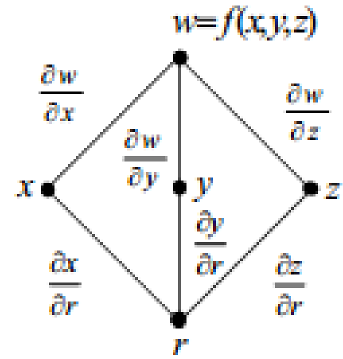
$$=(y)(-\sin(t))+(x)(\cos(t))+(1)(1)$$

$$=(\sin(t))(-\sin(t))+(\cos(t))(\cos(t))+1$$

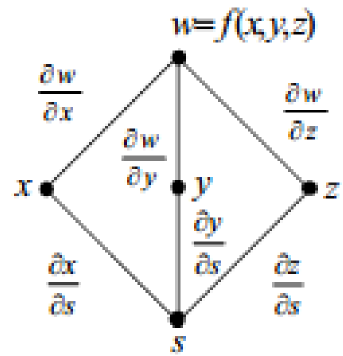
$$=-\sin^2(t)+\cos^2(t)+1=1+\cos(2t)$$

$$\left(\frac{dw}{dt}\right)_{t=0}=1+\cos(0)=2$$

Chain Rule for Two Independent Variables and Three Intermediate Variables



Suppose that $w=f(x,y,z)$, $x=g(r,s)$, $y=h(r,s)$, $z=k(r,s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas



$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example (1):

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w=x+2y+z^2, \qquad x=\frac{r}{s}, \qquad y=r^2+\ln s, \qquad z=2r$$

Solution

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial r}=(1)\left(\frac{1}{s}\right)+(2)(2r)+(2z)(2)$$

$$=\frac{1}{s}+4r+4(2r)=\frac{1}{s}+12r$$

$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=(1)\left(-\frac{r}{s^2}\right)+(2)\left(\frac{1}{s}\right)+(2z)(0)$$

$$=\frac{2}{s}-\frac{r}{s^2}$$

Example (2):

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s$$

Solution

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1)$$

$$= 2(r-s) + 2(r+s) \;\; = 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x)(-1) + (2y)(1)$$

$$= -2(r-s) + 2(r+s) \;\; = 4s$$

Example (3): If $z = x^n f\left(\frac{y}{x}\right)$, Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

$$x \cdot \frac{\partial z}{\partial x} = -x^{n-1} \cdot y\, f'(\frac{y}{x}) + n\, x^n \cdot f(\frac{y}{x}) \qquad \ldots\ldots (1)$$

$$\frac{\partial z}{\partial y} = x^n \cdot f'(\frac{y}{x}) \cdot (\frac{1}{x}) + 0$$

$$y \cdot \frac{\partial z}{\partial y} = x^{n-1} \cdot y\, f'(\frac{y}{x}) \qquad \ldots\ldots (2)$$

From (1) & (2)

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = nx^n f(\frac{y}{x})$$

$$= nz$$

Exercise

Express $\frac{\partial \omega}{\partial r}$ and $\frac{\partial \omega}{\partial s}$ in terms of r & s if $\omega = x + 2y + z^2$,

$x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

Ans: $\frac{\partial \omega}{\partial r} = 12r + \frac{1}{s}$, $\frac{\partial \omega}{\partial s} = \frac{-r}{s^2} + \frac{2}{s}$

H.W

1- Find

Find $\partial w / \partial u$ ***and*** $\partial w / \partial v$ ***for*** $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$ ***at the point*** $(u, v) = (1/2, 1)$.

2- Find *dw/dt* at the given value for the following function

$w = x^2 + y^2$, $x = \cos(t)$, $y = \sin(t)$, at $t = \pi$ ***Ans.*** $\left.\frac{dw}{dt}\right|_{t=\pi} = 0$

Maximum and Minima points (Exterme Values) and Saddle points

The extreme values of $f(x,y)$ can occur only at

i. ***Boundary points*** of the domain of f and ***endpoints***.

ii. ***Critical points*** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-partial derivatives of f are continuous throughout a point (a,b) and $f_x(a,b) = f_y(a,b) = 0$, the nature of $f(a,b)$ can be tested with the ***Second Derivative Test***:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) $\Rightarrow$ ***Local Maximum***

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b) $\Rightarrow$ ***Local Minimum***

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b) $\Rightarrow$ ***Saddle Point***

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b) $\Rightarrow$ ***Test is inconclusive***

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the discriminant of f and written in determinant form as follows:

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Ex.1: Locate M,m & S (if any)

$$f = x^2 - xy + y^2 + 2x + 2y - 4$$

$f_x = 2x - y + 2$

$f_y = -x + 2y + 2$

$2x - y + 2 = 0 \quad ...(1)$

$-x + 2y + 2 = 0 \quad ...(2)$

multi (1) by 2 + (2)

$\Rightarrow \quad 3x + 6 = 0 \quad \Rightarrow \quad x = -2 \quad , \quad y = -2 \quad , \quad (-2,-2)$

$f_{xx} = 2 \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = -1$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 = (2)(2) - 1 = 3 \quad \succ 0$

Since $f_{xx} \succ 0 \quad \Rightarrow \quad (-2,-2)$ is **m**

Ex.2: Locate M,m & S (if any)

$$f = x^3 + y^3 - 3axy$$

$f_x = 3x^2 - 3ay$

$f_y = 3y^2 - 3ax$

$3x^2 - 3ay = 0 \quad ...(1)$

$3y^2 - 3ax = 0 \quad ...(2)$

From (1) $\Rightarrow \quad y = \dfrac{x^2}{a}$

In (2) $\Rightarrow \quad \dfrac{x^4}{a^2} - ax = 0$

$\Rightarrow \quad x^4 - a^3x = 0$

$x(x^3 - a^3) = 0$

$\Rightarrow \quad x = 0 \quad , \quad x = a$

$\therefore \; y = 0 \quad , \quad y = a$

$\Rightarrow$ (0,0) & (a,a)

$f_{xx} = 6x \quad , \quad f_{yy} = 6y \quad , \quad f_{xy} = -3a$

1) $at\ (0,0) \Rightarrow f_{xx}=0 \quad , \quad f_{yy}=0 \quad , \quad f_{xy}=-3a$

$$f_{xx}\cdot f_{yy}-(f_{xy})^2=-9a^2 \quad \prec 0$$

$(0,0)$ is a saddle point

2) $at\ (a,a) \Rightarrow f_{xx}=6a \quad , \quad f_{yy}=6a \quad , \quad f_{xy}=-3a$

$$f_{xx}\cdot f_{yy}-(f_{xy})^2=(6a)(6a)-9a^2$$

$$=36a^2-9a^2=27a^2 \quad \succ 0$$

i) if $a \succ 0 \Rightarrow f_{xx} \succ 0 \Rightarrow (a,a)$ is **m**

ii) if $a \prec 0 \Rightarrow f_{xx} \prec 0 \Rightarrow (a,a)$ is **M**

EX.3 :

Find the local extreme values of the function

$$f(x,y)=xy-x^2-y^2-2x-2y+4$$

Solution

The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2,-2)$ is the only point where f may take on an extreme value.

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a,b) = (-2,-2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$$

The combination $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ tells us that f has a local maximum at $(-2,-2)$. The value of f at this point is $f(-2,-2) = 8$.

Homework

1) $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$ ***Ans.*** $f(2,-1) = -6$ local min.

2) $f(x,y) = x^2 - y^2 - 2x + 4y + 6$ ***Ans.*** $f(1,2)$ saddle point

3) $f(x,y) = x^2 + 2xy$ ***Ans.*** $f(0,0)$ saddle point