CHAPTER 12

Partial Differential Equations (PDEs)

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. Usually one of these deals with time t and the remaining with space (spatial variable(s)). The most important PDEs are the wave equations that can model the vibrating string (Secs. 12.2, 12.3, 12.4, 12.12) and the vibrating membrane (Secs. 12.8, 12.9, 12.10), the heat equation for temperature in a bar or wire (Secs. 12.5, 12.6), and the Laplace equation for electrostatic potentials (Secs. 12.6, 12.10, 12.11). PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics. They have a much wider range of applications than ODEs, which can model only the simplest physical systems. Thus PDEs are subjects of many ongoing research and development projects.

Realizing that modeling with PDEs is more involved than modeling with ODEs, we take a gradual, well-planned approach to modeling with PDEs. To do this we carefully derive the PDE that models the phenomena, such as the one-dimensional wave equation for a vibrating elastic string (say a violin string) in Sec. 12.2, and then solve the PDE in a separate section, that is, Sec. 12.3. In a similar vein, we derive the heat equation in Sec. 12.5 and then solve and generalize it in Sec. 12.6.

We derive these PDEs from physics and consider methods for solving initial and boundary value problems, that is, methods of obtaining solutions which satisfy the conditions required by the physical situations. In Secs. 12.7 and 12.12 we show how PDEs can also be solved by Fourier and Laplace transform methods.

COMMENT. ***Numerics for PDEs*** is explained in Secs. 21.4–21.7, which, for greater teaching flexibility, is designed to be independent of the other sections on numerics in Part E.

Prerequisites: Linear ODEs (Chap. 2), Fourier series (Chap. 11).
Sections that may be omitted in a shorter course: 12.7, 12.10–12.12.
References and Answers to Problems: App. 1 Part C, App. 2.

12.1 Basic Concepts of PDEs

A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an (unknown) function, call it u, that depends on two or more variables, often time t and one or several variables in space. The order of the highest derivative is called the **order** of the PDE. Just as was the case for ODEs, second-order PDEs will be the most important ones in applications.

Just as for ordinary differential equations (ODEs) we say that a PDE is **linear** if it is of the first degree in the unknown function u and its partial derivatives. Otherwise we call it **nonlinear**. Thus, all the equations in Example 1 are linear. We call a *linear* PDE **homogeneous** if each of its terms contains either u or one of its partial derivatives. Otherwise we call the equation **nonhomogeneous**. Thus, (4) in Example 1 (with f not identically zero) is nonhomogeneous, whereas the other equations are homogeneous.

EXAMPLE 1 Important Second-Order PDEs

(1) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ***One-dimensional wave equation***

(2) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ***One-dimensional heat equation***

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ ***Two-dimensional Laplace equation***

(4) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$ ***Two-dimensional Poisson equation***

(5) $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ ***Two-dimensional wave equation***

(6) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ ***Three-dimensional Laplace equation***

Here c is a positive constant, t is time, x, y, z are Cartesian coordinates, and *dimension* is the number of these coordinates in the equation. ■

A **solution** of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D (definition in Sec. 9.6) containing R, and satisfies the PDE everywhere in R.

Often one merely requires that the function is continuous on the boundary of R, has those derivatives in the interior of R, and satisfies the PDE in the interior of R. Letting R lie in D simplifies the situation regarding derivatives on the boundary of R, which is then the same on the boundary as it is in the interior of R.

In general, the totality of solutions of a PDE is very large. For example, the functions

(7) $$u = x^2 - y^2, \qquad u = e^x \cos y, \qquad u = \sin x \cosh y, \qquad u = \ln (x^2 + y^2)$$

which are entirely different from each other, are solutions of (3), as you may verify. We shall see later that the unique solution of a PDE corresponding to a given physical problem will be obtained by the use of ***additional conditions*** arising from the problem. For instance, this may be the condition that the solution u assume given values on the boundary of the region R ("**boundary conditions**"). Or, when time t is one of the variables, u (or $u_t = \partial u/\partial t$ or both) may be prescribed at $t = 0$ ("**initial conditions**").

We know that if an ODE is linear and homogeneous, then from known solutions we can obtain further solutions by superposition. For PDEs the situation is quite similar:

THEOREM 1 Fundamental Theorem on Superposition

*If u_1 and u_2 are solutions of a **homogeneous linear** PDE in some region R, then*

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 and c_2 is also a solution of that PDE in the region R.

The simple proof of this important theorem is quite similar to that of Theorem 1 in Sec. 2.1 and is left to the student.

Verification of solutions in Probs. 2–13 proceeds as for ODEs. Problems 16–23 concern PDEs solvable like ODEs. To help the student with them, we consider two typical examples.

EXAMPLE 2 **Solving $u_{xx} - u = 0$ Like an ODE**

Find solutions u of the PDE $u_{xx} - u = 0$ depending on x and y.

Solution. Since no y-derivatives occur, we can solve this PDE like $u'' - u = 0$. In Sec. 2.2 we would have obtained $u = Ae^{x} + Be^{-x}$ with constant A and B. Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)e^{x} + B(y)e^{-x}$$

with arbitrary functions A and B. We thus have a great variety of solutions. Check the result by differentiation. ■

EXAMPLE 3 **Solving $u_{xy} = -u_x$ Like an ODE**

Find solutions $u = u(x, y)$ of this PDE.

Solution. Setting $u_x = p$, we have $p_y = -p$, $p_y/p = -1$, $\ln|p| = -y + \tilde{c}(x)$, $p = c(x)e^{-y}$ and by integration with respect to x,

$$u(x, y) = f(x)e^{-y} + g(y) \qquad \text{where} \qquad f(x) = \int c(x)\,dx,$$

here, $f(x)$ and $g(y)$ are arbitrary. ■

PROBLEM SET 12.1

1. Fundamental theorem. Prove it for second-order PDEs in two and three independent variables. *Hint.* Prove it by substitution.

2–13 VERIFICATION OF SOLUTIONS

Verifiy (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

2–5 Wave Equation (1) with suitable c

2. $u = x^2 + t^2$

3. $u = \cos 4t \sin 2x$

4. $u = \sin kct \cos kx$

5. $u = \sin at \sin bx$

6–9 Heat Equation (2) with suitable c

6. $u = e^{-t} \sin x$

7. $u = e^{-\omega^2 c^2 t} \cos \omega x$

8. $u = e^{-9t} \sin \omega x$

9. $u = e^{-\pi^2 t} \cos 25x$

10–13 Laplace Equation (3)

10. $u = e^{x} \cos y,\ e^{x} \sin y$

11. $u = \arctan(y/x)$

12. $u = \cos y \sinh x,\ \sin y \cosh x$

13. $u = x/(x^2 + y^2),\ y/(x^2 + y^2)$

14. TEAM PROJECT. Verification of Solutions

(a) Wave equation. Verify that $u(x, t) = v(x + ct) + w(x - ct)$ with any twice differentiable functions v and w satisfies (1).

(b) Poisson equation. Verify that each u satisfies (4) with $f(x, y)$ as indicated.

$u = y/x$ $\qquad f = 2y/x^3$

$u = \sin xy$ $\qquad f = (x^2 + y^2) \sin xy$

$u = e^{x^2 - y^2}$ $\qquad f = 4(x^2 + y^2)e^{x^2 - y^2}$

$u = 1/\sqrt{x^2 + y^2}$ $\qquad f = (x^2 + y^2)^{-3/2}$

(c) Laplace equation. Verify that $u = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies (6) and $u = \ln(x^2 + y^2)$ satisfies (3). Is $u = 1/\sqrt{x^2 + y^2}$ a solution of (3)? Of what Poisson equation?

(d) Verify that u with any (sufficiently often differentiable) v and w satisfies the given PDE.

$u = v(x) + w(y)$ $\qquad u_{xy} = 0$

$u = v(x)w(y)$ $\qquad uu_{xy} = u_x u_y$

$u = v(x + 2t) + w(x - 2t)$ $\qquad u_{tt} = 4u_{xx}$

15. Boundary value problem. Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation

(3) and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

16–23 PDEs SOLVABLE AS ODEs

This happens if a PDE involves derivatives with respect to one variable only (or can be transformed to such a form), so that the other variable(s) can be treated as parameter(s). Solve for $u = u(x, y)$:

16. $u_{yy} = 0$

17. $u_{xx} + 16\pi^2 u = 0$

18. $25u_{yy} - 4u = 0$

19. $u_y + y^2 u = 0$

20. $2u_{xx} + 9u_x + 4u = -3 \cos x - 29 \sin x$

21. $u_{yy} + 6u_y + 13u = 4e^{3y}$

22. $u_{xy} = u_x$

23. $x^2 u_{xx} + 2xu_x - 2u = 0$

24. Surface of revolution. Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.

25. System of PDEs. Solve $u_{xx} = 0, u_{yy} = 0$

12.2 Modeling: Vibrating String, Wave Equation

In this section we model a vibrating string, which will lead to our first important PDE, that is, equation (3) which will then be solved in Sec. 12.3. *The student should pay very close attention to this delicate modeling process and detailed derivation starting from scratch,* as the skills learned can be applied to modeling other phenomena in general and in particular to modeling a vibrating membrane (Sec. 12.7).

We want to derive the PDE modeling small transverse vibrations of an elastic string, such as a violin string. We place the string along the x-axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$. We then distort the string, and at some instant, call it $t = 0$, we release it and allow it to vibrate. The problem is to determine the vibrations of the string, that is, to find its deflection $u(x, t)$ at any point x and at any time $t > 0$; see Fig. 286.

$u(x, t)$ will be the solution of a PDE that is the model of our physical system to be derived. This PDE should not be too complicated, so that we can solve it. Reasonable simplifying assumptions (just as for ODEs modeling vibrations in Chap. 2) are as follows.

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string"). The string is perfectly elastic and does not offer any resistance to bending.
2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value.

Under these assumptions we may expect solutions $u(x, t)$ that describe the physical reality sufficiently well.

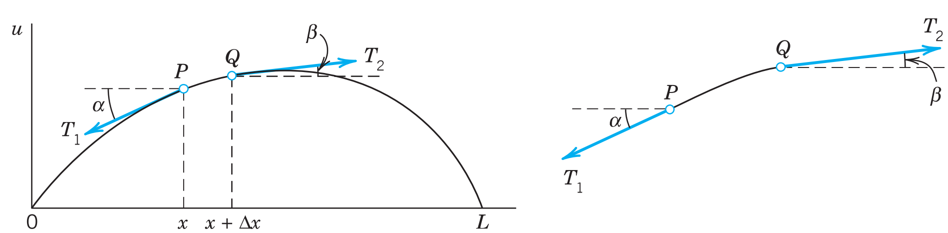

Fig. 286. Deflected string at fixed time t. Explanation on p. 544

Derivation of the PDE of the Model ("Wave Equation") from Forces

The model of the vibrating string will consist of a PDE ("wave equation") and additional conditions. To obtain the PDE, we consider the ***forces acting on a small portion of the string*** (Fig. 286). This method is typical of modeling in mechanics and elsewhere.

Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tension at the endpoints P and Q of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant. Using the notation shown in Fig. 286, we thus obtain

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{const.} \tag{1}$$

In the vertical direction we have two forces, namely, the vertical components $-T_1 \sin\alpha$ and $T_2 \sin\beta$ of T_1 and T_2; here the minus sign appears because the component at P is directed downward. By **Newton's second law** (Sec. 2.4) the resultant of these two forces is equal to the mass $\rho\Delta x$ of the portion times the acceleration $\partial^2 u/\partial t^2$, evaluated at some point between x and $x + \Delta x$; here ρ is the mass of the undeflected string per unit length, and Δx is the length of the portion of the undeflected string. (Δ is generally used to denote small quantities; this has nothing to do with the Laplacian ∇^2, which is sometimes also denoted by Δ.) Hence

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\Delta x \frac{\partial^2 u}{\partial t^2}.$$

Using (1), we can divide this by $T_2 \cos\beta = T_1 \cos\alpha = T$, obtaining

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \tag{2}$$

Now $\tan\alpha$ and $\tan\beta$ are the slopes of the string at x and $x + \Delta x$:

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)\bigg|_x \quad \text{and} \quad \tan\beta = \left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x}.$$

Here we have to write *partial* derivatives because u also depends on time t. Dividing (2) by Δx, we thus have

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\bigg|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\bigg|_x\right] = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}.$$

If we let Δx approach zero, we obtain the linear PDE

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},} \qquad c^2 = \frac{T}{\rho}. \tag{3}$$

This is called the **one-dimensional wave equation**. We see that it is homogeneous and of the second order. The physical constant T/ρ is denoted by c^2 (instead of c) to indicate that this constant is *positive,* a fact that will be essential to the form of the solutions. "One-dimensional" means that the equation involves only one space variable, x. In the next section we shall complete setting up the model and then show how to solve it by a general method that is probably the most important one for PDEs in engineering mathematics.

12.3 Solution by Separating Variables. Use of Fourier Series

We continue our work from Sec. 12.2, where we modeled a vibrating string and obtained the one-dimensional wave equation. We now have to complete the model by adding additional conditions and then solving the resulting model.

The model of a vibrating elastic string (a violin string, for instance) consists of the **one-dimensional wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho} \tag{1}$$

for the unknown deflection $u(x, t)$ of the string, a PDE that we have just obtained, and some ***additional conditions***, which we shall now derive.

Since the string is fastened at the ends $x = 0$ and $x = L$ (see Sec. 12.2), we have the two **boundary conditions**

$$\text{(a)} \quad u(0, t) = 0, \qquad \text{(b)} \quad u(L, t) = 0, \qquad \text{for all } t \geqq 0. \tag{2}$$

Furthermore, the form of the motion of the string will depend on its *initial deflection* (deflection at time $t = 0$), call it $f(x)$, and on its *initial velocity* (velocity at $t = 0$), call it $g(x)$. We thus have the two **initial conditions**

$$\text{(a)} \quad u(x, 0) = f(x), \qquad \text{(b)} \quad u_t(x, 0) = g(x) \qquad (0 \leqq x \leqq L) \tag{3}$$

where $u_t = \partial u/\partial t$. We now have to find a solution of the PDE (1) satisfying the conditions (2) and (3). This will be the solution of our problem. We shall do this in three steps, as follows.

Step 1. By the "**method of separating variables**" or *product method,* setting $u(x, t) = F(x)G(t)$, we obtain from (1) two ODEs, one for $F(x)$ and the other one for $G(t)$.

Step 2. We determine solutions of these ODEs that satisfy the boundary conditions (2).

Step 3. Finally, using **Fourier series**, we compose the solutions found in Step 2 to obtain a solution of (1) satisfying both (2) and (3), that is, the solution of our model of the vibrating string.

Step 1. Two ODEs from the Wave Equation (1)

In the **method of separating variables**, or *product method*, we determine solutions of the wave equation (1) of the form

$$u(x, t) = F(x)G(t) \tag{4}$$

which are a product of two functions, each depending on only one of the variables x and t. This is a powerful general method that has various applications in engineering mathematics, as we shall see in this chapter. Differentiating (4), we obtain

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = F''G$$

where dots denote derivatives with respect to t and primes derivatives with respect to x. By inserting this into the wave equation (1) we have

$$F\ddot{G} = c^2 F''G.$$

Dividing by c^2FG and simplifying gives

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}.$$

The variables are now separated, the left side depending only on t and the right side only on x. Hence both sides must be constant because, if they were variable, then changing t or x would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k.$$

Multiplying by the denominators gives immediately two ***ordinary*** DEs

(5) $$F'' - kF = 0$$

and

(6) $$\ddot{G} - c^2 kG = 0.$$

Here, the **separation constant** k is still arbitrary.

Step 2. Satisfying the Boundary Conditions (2)

We now determine solutions F and G of (5) and (6) so that $u = FG$ satisfies the boundary conditions (2), that is,

(7) $$u(0, t) = F(0)G(t) = 0, \qquad u(L, t) = F(L)G(t) = 0 \qquad \text{for all } t.$$

We first solve (5). If $G \equiv 0$, then $u = FG \equiv 0$, which is of no interest. Hence $G \not\equiv 0$ and then by (7),

(8) $$\text{(a)} \quad F(0) = 0, \qquad \text{(b)} \quad F(L) = 0.$$

We show that k must be negative. For $k = 0$ the general solution of (5) is $F = ax + b$, and from (8) we obtain $a = b = 0$, so that $F \equiv 0$ and $u = FG \equiv 0$, which is of no interest. For positive $k = \mu^2$ a general solution of (5) is

$$F = Ae^{\mu x} + Be^{-\mu x}$$

and from (8) we obtain $F \equiv 0$ as before (verify!). Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then (5) becomes $F'' + p^2 F = 0$ and has as a general solution

$$F(x) = A \cos px + B \sin px.$$

From this and (8) we have

$$F(0) = A = 0 \qquad \text{and then} \qquad F(L) = B \sin pL = 0.$$

We must take $B \neq 0$ since otherwise $F \equiv 0$. Hence $\sin pL = 0$. Thus

$$pL = n\pi, \qquad \text{so that} \qquad p = \frac{n\pi}{L} \qquad (n \text{ integer}). \tag{9}$$

Setting $B = 1$, we thus obtain infinitely many solutions $F(x) = F_n(x)$, where

$$F_n(x) = \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots). \tag{10}$$

These solutions satisfy (8). [For negative integer n we obtain essentially the same solutions, except for a minus sign, because $\sin(-\alpha) = -\sin \alpha$.]

We now solve (6) with $k = -p^2 = -(n\pi/L)^2$ resulting from (9), that is,

$$\ddot{G} + \lambda_n^2 G = 0 \qquad \text{where} \qquad \lambda_n = cp = \frac{cn\pi}{L}. \tag{11*}$$

A general solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of (1) satisfying (2) are $u_n(x, t) = F_n(x) G_n(t) = G_n(t) F_n(x)$, written out

$$u_n(x, t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \qquad (n = 1, 2, \cdots). \tag{11}$$

These functions are called the **eigenfunctions**, or *characteristic functions*, and the values $\lambda_n = cn\pi/L$ are called the **eigenvalues**, or *characteristic values*, of the vibrating string. The set $\{\lambda_1, \lambda_2, \cdots\}$ is called the **spectrum**.

Discussion of Eigenfunctions. We see that each u_n represents a harmonic motion having the **frequency** $\lambda_n/2\pi = cn/2L$ cycles per unit time. This motion is called the nth **normal mode** of the string. The first normal mode is known as the *fundamental mode* ($n = 1$), and the others are known as *overtones;* musically they give the octave, octave plus fifth, etc. Since in (11)

$$\sin \frac{n\pi x}{L} = 0 \qquad \text{at} \qquad x = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n} L,$$

the nth normal mode has $n - 1$ **nodes**, that is, points of the string that do not move (in addition to the fixed endpoints); see Fig. 287.

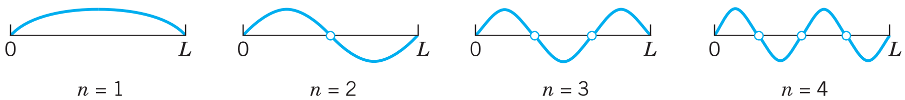

Fig. 287. Normal modes of the vibrating string

Figure 288 shows the second normal mode for various values of t. At any instant the string has the form of a sine wave. When the left part of the string is moving down, the other half is moving up, and conversely. For the other modes the situation is similar.

Tuning is done by changing the tension T. Our formula for the frequency $\lambda_n/2\pi = cn/2L$ of u_n with $c = \sqrt{T/\rho}$ [see (3), Sec. 12.2] confirms that effect because it shows that the frequency is proportional to the tension. T cannot be increased indefinitely, but can you see what to do to get a string with a high fundamental mode? (Think of both L and ρ.) Why is a violin smaller than a double-bass?

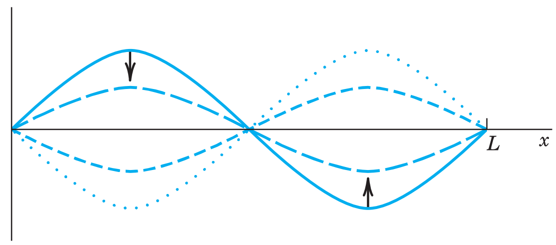

Fig. 288. Second normal mode for various values of t

Step 3. Solution of the Entire Problem. Fourier Series

The eigenfunctions (11) satisfy the wave equation (1) and the boundary conditions (2) (string fixed at the ends). A single u_n will generally not satisfy the initial conditions (3). But since the wave equation (1) is linear and homogeneous, it follows from Fundamental Theorem 1 in Sec. 12.1 that the sum of finitely many solutions u_n is a solution of (1). To obtain a solution that also satisfies the initial conditions (3), we consider the infinite series (with $\lambda_n = cn\pi/L$ as before)

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{12}$$

Satisfying Initial Condition (3a) (Given Initial Displacement). From (12) and (3a) we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x). \qquad (0 \leqq x \leqq L). \tag{13}$$

Hence we must choose the B_n's so that $u(x, 0)$ becomes the **Fourier sine series** of $f(x)$. Thus, by (4) in Sec. 11.3,

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, \cdots. \tag{14}$$

Satisfying Initial Condition (3b) (Given Initial Velocity). Similarly, by differentiating (12) with respect to t and using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}(-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n \cos\lambda_n t)\sin\frac{n\pi x}{L}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi x}{L} = g(x).$$

Hence we must choose the B_n^*'s so that for $t = 0$ the derivative $\partial u/\partial t$ becomes the Fourier sine series of $g(x)$. Thus, again by (4) in Sec. 11.3,

$$B_n^*\lambda_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Since $\lambda_n = cn\pi/L$, we obtain by division

(15) $$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

Result. Our discussion shows that $u(x, t)$ given by (12) with coefficients (14) and (15) is a solution of (1) that satisfies all the conditions in (2) and (3), provided the series (12) converges and so do the series obtained by differentiating (12) twice termwise with respect to x and t and have the sums $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial t^2$, respectively, which are continuous.

Solution (12) Established. According to our derivation, the solution (12) is at first a purely formal expression, but we shall now establish it. For the sake of simplicity we consider only the case when the initial velocity $g(x)$ is identically zero. Then the B_n^* are zero, and (12) reduces to

(16) $$u(x, t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L}.$$

It is possible to ***sum this series***, that is, to write the result in a closed or finite form. For this purpose we use the formula [see (11), App. A3.1]

$$\cos\frac{cn\pi}{L}t\sin\frac{n\pi}{L}x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right].$$

Consequently, we may write (16) in the form

$$u(x, t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}.$$

These two series are those obtained by substituting $x - ct$ and $x + ct$, respectively, for the variable x in the Fourier sine series (13) for $f(x)$. Thus

(17) $$u(x, t) = \tfrac{1}{2}[f^*(x-ct) + f^*(x+ct)]$$

where f^* is the odd periodic extension of f with the period $2L$ (Fig. 289). Since the initial deflection $f(x)$ is continuous on the interval $0 \leqq x \leqq L$ and zero at the endpoints, it follows from (17) that $u(x, t)$ is a continuous function of both variables x and t for all values of the variables. By differentiating (17) we see that $u(x, t)$ is a solution of (1), provided $f(x)$ is twice differentiable on the interval $0 < x < L$, and has one-sided second derivatives at $x = 0$ and $x = L$, which are zero. Under these conditions $u(x, t)$ is established as a solution of (1), satisfying (2) and (3) with $g(x) \equiv 0$. ■

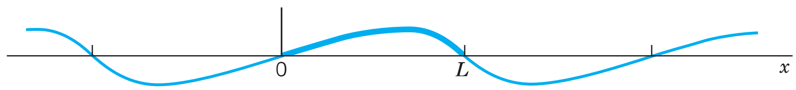

Fig. 289. Odd periodic extension of $f(x)$

Generalized Solution. If $f'(x)$ and $f''(x)$ are merely piecewise continuous (see Sec. 6.1), or if those one-sided derivatives are not zero, then for each t there will be finitely many values of x at which the second derivatives of u appearing in (1) do not exist. Except at these points the wave equation will still be satisfied. We may then regard $u(x, t)$ as a "**generalized solution**," as it is called, that is, as a solution in a broader sense. For instance, a triangular initial deflection as in Example 1 (below) leads to a generalized solution.

Physical Interpretation of the Solution (17). The graph of $f^*(x - ct)$ is obtained from the graph of $f^*(x)$ by shifting the latter ct units to the right (Fig. 290). This means that $f^*(x - ct)(c > 0)$ represents a wave that is traveling to the right as t increases. Similarly, $f^*(x + ct)$ represents a wave that is traveling to the left, and $u(x, t)$ is the superposition of these two waves.

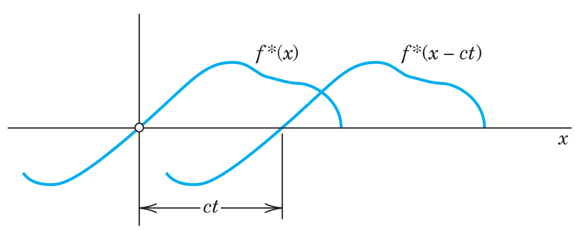

Fig. 290. Interpretation of (17)

EXAMPLE 1 Vibrating String if the Initial Deflection Is Triangular

Find the solution of the wave equation (1) satisfying (2) and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\[2ex] \dfrac{2k}{L}(L - x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases}$$

and initial velocity zero. (Figure 291 shows $f(x) = u(x, 0)$ at the top.)

Solution. Since $g(x) \equiv 0$, we have $B_n^* = 0$ in (12), and from Example 4 in Sec. 11.3 we see that the B_n are given by (5), Sec. 11.3. Thus (12) takes the form

$$u(x, t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right].$$