Wavelets and Multiresolution Processing

1. Introduction

Wavelets are functions generated from one single function (basis function) called the prototype or **mother wavelet** by dilations (scaling) and translations (shifts) in time (frequency) domain.

Unlike the Fourier transform, which decomposes a signal to a sum of sinusoids, the wavelet transform decomposes a signal (image) to **small waves** of varying frequency and limited duration. The advantage is that we also know when (where) the frequency appear. Many applications in image compression, transmission, and analysis. We will examine wavelets from a multiresolution point of view and begin with an overview of imaging techniques involved in multiresolution theory.

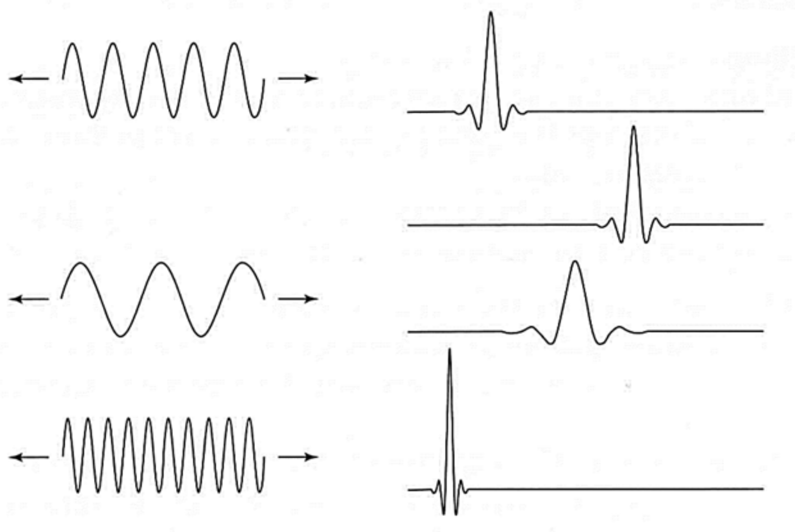

Figure 1: Fourier Transform and Wavelet

Wavelet transform is used to analyse a signal into different frequency components at different resolution scales (i.e. multiresolution). This allows revealing image's spatial and frequency attributes simultaneously. In addition, features that might go undetected at one resolution may be easy to spot at another. Multiresolution theory incorporates image pyramid and sub band coding techniques.

Wavelet analysis is computed by applying a series of Low and High pass on the signal, as shown in **Figure 2**.

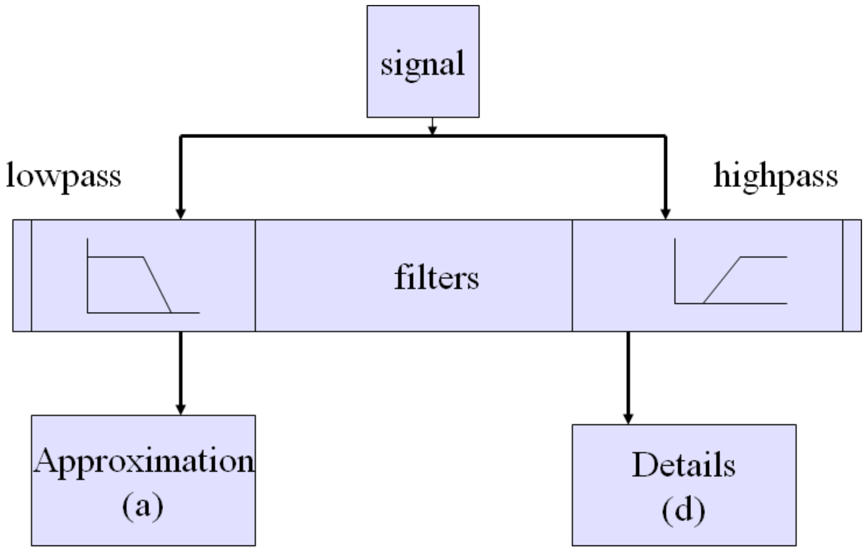

Figure 2: Wavelet analysis

2. Fourier Transform and Wavelet

- FFT, basis functions: sinusoids
- Wavelet transforms: small waves, called wavelet
- FFT can only offer frequency information
- Wavelet: frequency + temporal information
- Fourier analysis doesn't work well on discontinuous, "bursty" data music, video, power, earthquakes, …
- **Fourier**
  - Loses time (location) coordinate completely
  - Analyses the whole signal
  - Short pieces lose "frequency" meaning
- **Wavelets**
  - Localized time-frequency analysis
  - Short signal pieces also have significance
  - Scale = Frequency band

3. Sub-Band coding

- A Source output is decomposed into its elements.
- It separates the source output into bands of different frequency using digital filters.
- Different filters are used like low pass filter or high pass filter.
- An image is decomposed to a set of bandlimited components (sub bands).
- The decomposition is carried by filtering and down sampling (Analysis step)
- If the filters are properly selected the image may be reconstructed without error by filtering and up sampling (synthesis Step)
- As shown in **Figure 3**, One dimension analysis and synthesis steps. $\boldsymbol{f(n)}$ is the input signal,

h_0 *and* h_1 is the low and high analysis filter (forward Discrete Wavelet Transform (DWT)), g_0 *and* g_1 is the low and high synthesis filter (Inverse Discrete Wavelet Transform (IDWT)). **f_{lp} represents the low pass filtered version of the signal (Approximation), f_{hp} represents the High Pass Filtered version of the signal (Details),**

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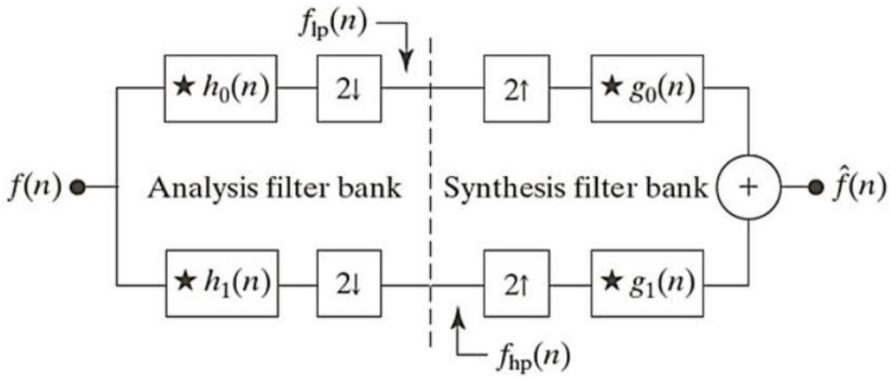

Figure 3: One Dimension Analysis and synthesis steps (Forward and Inverse Wavelet Transform)

4. Discrete Haar Wavelet Transform (DHWT)

There are many discrete wavelet transforms they are Coiflet, Daubechies, Haar, Symmlet, …. The Basic filter is the Haar. The differences between them are in the filter size, frequency response and hardware complexity. In this lecture we will focus on the Basic Wavelet Filter bank (Haar wavelet).

The Haar wavelet is the first known wavelet. The Haar wavelet is also the simplest possible wavelet.

Each step in the one dimensional Haar wavelet transform calculates a set of wavelet approximation (Lo-D) and a set of details (Hi-D). for $N-sample$ input signal the approximation a_m and details d_m can be computed from the following two equations:

$$a_m = \frac{f_{2m} + f_{2m-1}}{\sqrt{2}}, \qquad m = 1, 2, 3 \ldots \ldots N/2$$

$$d_m = \frac{f_{2m} - f_{2m-1}}{\sqrt{2}}, \qquad m = 1, 2, 3 \ldots \ldots N/2$$

Some references in the literature use 2 $rather\ than$ $\sqrt{2}$ in the above equations

The wavelet transformation can be written in matrix form as follows:

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example1: Find Two-level 1-D DHWT components for the following signal

f=[3 2 6 4 8 6 2 5]

Solution:

The First Level $(\boldsymbol{a_{mL1}}, \boldsymbol{d_{mL1}})$

$$a_{mL1} = \frac{2+3}{\sqrt{2}}, \frac{4+6}{\sqrt{2}}, \frac{6+8}{\sqrt{2}}, \frac{5+2}{\sqrt{2}} \; first\ leve\ LPF\ components$$

$$a_{mL1} = \frac{5}{\sqrt{2}}, \frac{10}{\sqrt{2}}, \frac{14}{\sqrt{2}}, \frac{7}{\sqrt{2}} \; first\ leve\ LPF\ components$$

$$d_{mL1} = \frac{2-3}{\sqrt{2}}, \frac{4-6}{\sqrt{2}}, \frac{6-8}{\sqrt{2}}, \frac{5-2}{\sqrt{2}} \; first\ leve\ HPF\ components$$

$$d_{mL1} = \frac{-1}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{3}{\sqrt{2}} \; first\ leve\ HPF\ components$$

The Second Level: $(\boldsymbol{a_{mL2}}, \boldsymbol{d_{mL2}})$

$$a_{mL2} = \frac{1}{\sqrt{2}}\left(\frac{10}{\sqrt{2}} + \frac{5}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\left(\frac{7}{\sqrt{2}} + \frac{14}{\sqrt{2}}\right) \; Second\ leve\ LPF\ components$$

$$a_{mL2} = \frac{1}{\sqrt{2}}\left(\frac{15}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\left(\frac{21}{\sqrt{2}}\right) \; Second\ leve\ LPF\ components$$

$$a_{mL2} = \frac{15}{2}, \frac{21}{2} \; Second\ leve\ LPF\ components$$

$$d_{mL2} = \frac{1}{\sqrt{2}}\left(\frac{10}{\sqrt{2}} - \frac{5}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\left(\frac{7}{\sqrt{2}} - \frac{14}{\sqrt{2}}\right) \; Second\ leve\ HPF\ components$$

$$d_{mL2} = \frac{1}{\sqrt{2}}\left(\frac{5}{\sqrt{2}}\right), \frac{1}{\sqrt{2}}\left(\frac{-7}{\sqrt{2}}\right) \; Second\ leve\ HPF\ components$$

$$d_{mL2} = \frac{5}{2}, \frac{-7}{2} \; Second\ leve\ HPF\ components$$

5. Inverse DHWT

To find the original signal from the decomposition component a synthesis filter (inverse) is applied. Inverse Haar wavelet is given by

$$\boldsymbol{f = \frac{1}{\sqrt{2}}(a_1 - d_1, a_1 + d_1, a_2 - d_2, a_2 + d_2, \ldots \ldots, a_{N/2} - d_{N/2}, a_{N/2} + d_{N/2})}$$

The computation steps should start from the last decomposition level to reconstruct the original image.

Example 2: Reconstruct the original signal of Example 1 using Inverse DHWT

Solution:

- **Start from Level2:**

$$a_{mL2} = \frac{15}{2}\ ,\frac{21}{2}\ \ Second\ leve\ LPF\ components\ --From\ Example\ 1$$

$$d_{mL2} = \frac{5}{2}\ ,\frac{-7}{2}\ \ Second\ leve\ HPF\ components\ \ --From\ Example\ 1$$

$$\boldsymbol{a_{mL1}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\frac{15}{2}-\frac{5}{2},\frac{15}{2}+\frac{5}{2},\frac{21}{2}-\frac{-7}{2},\frac{21}{2}+\frac{-7}{2}\right)$$

$$\boldsymbol{a_{mL1}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\frac{10}{2},\frac{20}{2},\frac{28}{2},\frac{14}{2}\right) \rightarrow \boldsymbol{a_{mL1}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(5, 10,14, 7\ )$$

The above values represent the first level Low Pass Filter Components
Start from Level2:
Retrieve the High Pass band from example 1

$$d_{mL1} = \frac{-1}{\sqrt{2}},\frac{-2}{\sqrt{2}}\ ,\frac{-2}{\sqrt{2}},\frac{3}{\sqrt{2}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(-1, -2, -2, 3\ )$$

Then the original function can be calculated as follows:

$$\boldsymbol{f} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\frac{5}{\sqrt{\mathbf{2}}}-\frac{-1}{\sqrt{\mathbf{2}}},\frac{5}{\sqrt{\mathbf{2}}}+\frac{-1}{\sqrt{\mathbf{2}}},\frac{10}{\sqrt{\mathbf{2}}}-\frac{-2}{\sqrt{\mathbf{2}}},\frac{10}{\sqrt{\mathbf{2}}}+\frac{-2}{\sqrt{\mathbf{2}}},\frac{14}{\sqrt{\mathbf{2}}}-\frac{-2}{\sqrt{\mathbf{2}}},\frac{14}{\sqrt{\mathbf{2}}}+\frac{-2}{\sqrt{\mathbf{2}}},\frac{7}{\sqrt{\mathbf{2}}}-\frac{3}{\sqrt{\mathbf{2}}},\frac{7}{\sqrt{\mathbf{2}}}+\frac{3}{\sqrt{\mathbf{2}}}\right)$$

$$\boldsymbol{f} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(\frac{6}{\sqrt{\mathbf{2}}},\frac{4}{\sqrt{\mathbf{2}}},\frac{12}{\sqrt{\mathbf{2}}},\frac{8}{\sqrt{\mathbf{2}}},\frac{16}{\sqrt{\mathbf{2}}},\frac{12}{\sqrt{\mathbf{2}}},\frac{4}{\sqrt{\mathbf{2}}},\frac{10}{\sqrt{\mathbf{2}}}\right)$$

Then the original signal is the same signal in Example 1 (reconstructed correctly)

f **=[3 2 6 4 8 6 2 5]**