3 - Linear First Order Equations:

A differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where P and Q are functions of x, is called a ***Linear First Order Equation***.

Find integrating factor $(I.f.) = e^{\int Pdx}$, then the general solution is

$$y \cdot (I.f.) = \int (I.f.)\ Q.\,dx$$

Ex.1: Solve $\frac{dy}{dx} - \frac{y}{x} = x \cdot e^x$

$P(x) = -\frac{1}{x}, \quad Q(x) = x.e^x$

$(I.f.) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

Solution is

$$y \cdot \frac{1}{x} = \int \frac{1}{x} \cdot xe^x \cdot dx$$

$$\frac{y}{x} = e^x + c$$

Ex .(2):

Solve the equation $(1+x^2)dy + (y - \tan^{-1}(x))dx = 0$.

Solution

Dividing the two sides by $(1+x^2)dx$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}(x)}{1+x^2} = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1}(x)}{1+x^2} \quad \Rightarrow \quad P(x) = \frac{1}{1+x^2}, \quad Q = \frac{\tan^{-1}(x)}{1+x^2}$$

$$\int P(x)dx = \int \frac{1}{1+x^2} dx = \tan^{-1}(x)$$

$$\rho(x) = e^{\tan^{-1}(x)}$$

$$e^{\tan^{-1}(x)} y = \int e^{\tan^{-1}(x)} \frac{\tan^{-1}(x)}{1+x^2} dx + C$$

$$z = \tan^{-1}(x) \quad \Rightarrow \quad dz = \frac{1}{1+x^2}dx$$

$$e^{\tan^{-1}(x)}y = \int e^z \times z dz + C$$

$$= ze^z - \int e^z dz + C$$

$$= ze^z - e^z + C$$

$$e^{\tan^{-1}(x)}y = \tan^{-1}(x)e^{\tan^{-1}(x)} - e^{\tan^{-1}(x)} + C$$

H.W:

1) $\frac{dy}{dx} + 2y = e^{-x}$ ans: $y = e^{-x} + ce^{-2x}$
2) $x\frac{dy}{dx} - 3y = x^2$ ans: $y = cx^3 - x^3$
3) $x\frac{dy}{dx} + 3y = \frac{\sin(x)}{x^2}$ ans: $x^3y = c - \cos(x)$
4) $xdy + ydx = ydy$ ans: $x = \frac{y}{2} + \frac{c}{y}$
5) $y' - y = 3$ ans: $y = -3 + ce^x$
6) $y' + 2y = 6e^x$ ans: $y =$
7) $y' + 2y = \cos(x)$ ans: $y =$
8) $y' + y = \sin(x)$ ans: $y =$

4- Exact Test ,Exact Solution:

A differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be ***exact*** if for some function $f(x,y)$

$$M(x,y)dx+N(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=df$$

is exact if and only if $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Example

- The equation $(x^2+y^2)dx+(2xy+\cos(y))dy=0$ is exact because the partial derivatives

  $$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)=2y, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(2xy+\cos(y))=2y$$

  are equal.

- The equation $(x+3y)dx+(x^2+\cos(y))dy=0$ is not exact because the partial derivatives

  $$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(x+3y)=3, \qquad \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}(x^2+\cos(y))=2x$$

  are not equal.

Solution of Exact Differential Equation:

i. Match the equation to the form $M(x,y)dx+N(x,y)dy=0$ to identify M and N.
ii. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$).
iii. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $g'(y)$ (or $g'(x)$).
iv. Integrate to find $g(y)$ (or $g(x)$).
v. Write the solution of the exact equation as $f(x,y)=C$.

Example

Solve the differential equation

$$(x^2 + y^2)dx + (2xy + \cos(y))dy = 0.$$

Solution

Step 1: ***Match the equation to the form*** $M(x, y)dx + N(x, y)dy = 0$ ***to identify*** M.

$$M(x, y) = x^2 + y^2$$

Step 2: ***Integrate*** M ***with respect to*** x, ***writing the constant of integration as*** $g(y)$.

$$f(x, y) = \int M(x, y)dx = \int (x^2 + y^2)dx = \frac{x^3}{3} + xy^2 + g(y)$$

Step 3: ***Differentiate with respect to*** y ***and set the result equal to*** N ***to find*** $g'(y)$.

$$\frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy^2 + g(y)\right) = 2xy + g'(y)$$

$$2xy + g'(y) = 2xy + \cos(y) \quad \Rightarrow \quad g'(y) = \cos(y)$$

Step 4: ***Integrate to find*** $g(y)$.

$$\int g'(y)dy = \int \cos(y)dy = \sin(y)$$

Step 5: ***Write the solution of the exact equation as*** $f(x, y) = C$.

$$\frac{x^3}{3} + xy^2 + \sin(y) = C$$

Homework

1) $(2x + e^y)dx + xe^y dy = 0$ $ans: f(x,y) = x^2 + e^y x + 1 = c$
2) $ydx + xdy = 0$ $ans: f(x,y) = \ yx + 1 = c$
3) $\frac{xdy - ydx}{x^2} = 0$ $ans: not\ exact\ D.E$
4) $(2xy + y^2)dx + (x^2 + 2xy - y)dy = 0$ $ans: x^2y + y^2x - \frac{y^2}{2} = c$
5) $(2 + ye^{xy})dx + (xe^{xy} - 2y)dy = 0$ $ans: 2x + e^{xy} - y^2 = c$
6) $(1 + x^2)dy + 2xydx = 0$ $ans:$
7) $ydx + x(1 + y)dy = 0$ $ans: 2x + e^{xy} - y^2 = c$

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