3- B. Second Order Differential Equations:

The second order linear differential equations with constant coefficient has the genral form is:

$ay'' + by' + cy = F(x) \quad \ldots(1),$

where a, b and c are constants.

If $F(x) = 0$ then (1) is called homogenous.

If $F(x) \neq 0$ then (1) is called non homogenous.

Linear Differential Operator

It is convenient to introduce the symbol D to represent the operation of differentiation with respect to x. That is, we write $Df(x)$ to mean df/dx. Furthermore, we define powers of D to mean taking successive derivatives:

$$D^2 f(x) = D\{Df(x)\} = \frac{d^2 f}{dx^2}, \qquad D^3 f(x) = D\{D^2 f(x)\} = \frac{d^3 f}{dx^3}$$

$$(D^2 + D - 2) f(x) = D^2 f(x) + Df(x) - 2f(x) = \frac{d^2 f}{dx^2} + \frac{df}{dx} - 2f(x)$$

a) The Second order linear homogenous D.Eq. with constant coefficient:

The general form is

$$ay'' + by' + cy = 0 \quad \ldots(2)$$

where a, b and c are constants.

The general solution

Put y'=Dy and y''=D^2y in eq. (2) (D is an operator)

$\Rightarrow$ a D^2y+bDy+cy=0

$\Rightarrow (aD^2 + bD + c)y = 0$ (using D-operator)

now substitute D by r and leave y then

$$ar^2 + br + c = 0$$

is called characteristic equation of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are two values of r :

1- ***real (equal and not equal).***
2- ***complex.***

Solution of $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0$	
Roots r_1 ***&*** r_2	***Solution***
Real and unequal	$y = C_1e^{r_1x} + C_2e^{r_2x}$
Real and equal	$y = (C_1x + C_2)e^{r_2x}$
Complex conjugate, $\alpha \pm j\beta$	$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0 \quad , \quad y = 1 \quad , \quad y' = r \quad , \quad y'' = r^2$

$(r+1)(r-3) = 0$

$r + 1 = 0 \quad \Rightarrow \quad r = -1$

$r - 3 = 0 \quad \Rightarrow \quad r = 3$

the general solution is

$y = c_1e^{-x} + c_2e^{3x}$

<u>Example</u>

Solve the following differential equations:

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, (b) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0$, (d) $\frac{d^2y}{dx^2} + 4y = 0$

Solution:

(a) $$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

The characteristic equation is

$$D^2+D-2=0$$

$$(D-1)(D+2)=0 \quad \Rightarrow \quad r_1=1 \quad \text{and} \quad r_2=-2$$

The solution is

$$y=C_1e^x+C_2e^{-2x}$$

(b) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

The characteristic equation is

$$D^2+4D+4=0$$

$$(D+2)^2=0 \quad \Rightarrow \quad r_1=r_2=-2$$

The solution is

$$y=(C_1x+C_2)e^{-2x}$$

(c) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$$

The characteristic equation is

$$D^2+4D+6=0$$

$$r_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$r_{1,2}=\frac{-4\pm\sqrt{(4)^2-4(1)(6)}}{2(1)}=\frac{-4\pm\sqrt{16-24}}{2}$$

$$r_{1,2}=\frac{-4\pm\sqrt{-8}}{2}=\frac{-4\pm j2\sqrt{2}}{2}$$

$$r_{1,2}=-2\pm j\sqrt{2} \quad \Rightarrow \quad r_1=-2+j\sqrt{2} \quad \text{and} \quad r_2=-2-j\surd$$

$$\Rightarrow \quad \alpha=-2 \quad \text{and} \quad \beta=\sqrt{2}$$

The solution is

$$y=e^{-2x}(C_1\cos\sqrt{2}x \quad + \quad C_2\sin\sqrt{2}x)$$

(d) $$\frac{d^2y}{dx^2} + 4y = 0$$

The characteristic equation is

$$D^2 + 4 = 0$$

$$(D - j2)(D + j2) = 0 \quad \Rightarrow \quad r_1 = j2 \quad \text{and} \quad r_2 = -j2$$

$$\Rightarrow \quad \alpha = 0 \quad \text{and} \quad \beta = 2$$

The solution is

$$y = C_1 \cos 2x \quad + \quad C_2 \sin 2x$$

Homework (1)

1. 4y''-12y'+5y=0 ans:$y=c_1e^{(1/2)x}+ c_2e^{(5/2)x}$
2. 3y''-14y'-5y=0 ans:$y=c_1e^{5x}+ c_2e^{(-1/3)x}$
3. 4y''+y=0 ans:$y=c_1\cos(x/2)+ c_2\sin(x/2)$
4. y''-8y'+16y=0 ans:$y=c_1e^{4x}+ c_2xe^{4x}$
5. y''+9y=0 ans:$y=c_1\cos3x+ c_2\sin3x$

b) The Second order linear non homogenous D.Eq. with constant coefficient:

The general form is: $ay'' + by' + cy = F(x)$ …(3)

where a, b and c are constants.

<u>The general solution</u>

If y_h is the solution of the homo. D.Eq. $ay'' + by' + cy = 0$, then the general solution of eq. (3) is:

$y = y_h + y_p$ y_h (complementary function)

y_p (porticular integral)

i) y_h is y homo.

ii) y_p

<u>Variation of Parameters to Find(y_p) :</u>

This method assumes we already know the homogeneous solution

$$y_h = C_1 u_1(x) + C_2 u_2(x)$$

The method consists of replacing the constants C_1 and C_2 by functions $v_1(x)$ and $v_2(x)$ and then requiring that the new expression

$$y_h = v_1 u_1 + v_2 u_2$$

and by solving the following two equations

$$v_1' u_1 + v_2' u_2 = 0$$

$$v_1' u_1' + v_2' u_2' = F(x)$$

for the unknown functions v_1' and v_2' using the following matrix notation

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ F(x) \end{bmatrix}$$

Finally v_1 and v_2 can be found by integration.

In applying the method of ***variation of parameters*** to find the particular solution, the following steps are taken:

i. Find v_1' and v_2' using the following equations

$$v_1' = \frac{\begin{vmatrix} 0 & u_2 \\ F(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{-u_2 F(x)}{D}, \qquad v_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & F(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 F(x)}{D}$$

where $$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

ii. Integrate v_1' and v_2' to find v_1 and v_2.

iii. Write the particular solution as

$$y_p = v_1 u_1 + v_2 u_2$$

<u>Example</u>

Solve the equation $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 6$

<u>Solution</u>

The homogeneous solution y_h can be found using the reduced equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

The characteristic equation is $D^2 + 2D - 3 = 0$ and the roots of this equation are $r_1 = -3$ and $r_2 = 1$, so

$$y_h = C_1 e^{-3x} + C_2 e^{x}$$

Then $$u_1 = e^{-3x},\ u_2 = e^{x}$$

$$D = \begin{vmatrix} e^{-3x} & e^{x} \\ -3e^{-3x} & e^{x} \end{vmatrix} = e^{-2x} + 3e^{-2x} = 4e^{-2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{x} \\ 6 & e^{x} \end{vmatrix}}{4e^{-2x}} = \frac{-6e^{x}}{4e^{-2x}} = -\frac{3}{2}e^{3x}, \qquad v_2' = \frac{\begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & 6 \end{vmatrix}}{4e^{-2x}} = \frac{6e^{-3x}}{4e^{-2x}} = \frac{3}{2}e^{-x}$$

$$v_1 = \int -\frac{3}{2}e^{3x}dx = -\frac{1}{2}e^{3x}, \qquad v_2 = \int \frac{3}{2}e^{-x}dx = -\frac{3}{2}e^{-x}$$

$$y_p = v_1 u_1 + v_2 u_2 = \left(-\frac{1}{2}e^{3x}\right)e^{-3x} + \left(-\frac{3}{2}e^{-x}\right)e^{x} = -2$$

$$y = y_h + y_p = C_1 e^{-3x} + C_2 e^{x} - 2$$

Example

Solve the equation $y'' - 2y' + y = e^{x}\ln(x)$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$D^2 - 2D + 1 = 0$$

$$(D-1)^2 = 0$$

The roots are $r_1 = r_2 = 1$

The solution is $y_h = (C_1 x + C_2)e^x$

$$y_h = C_1 x e^x + C_2 e^x$$

From that we have $u_1(x) = xe^x$, and $u_2(x) = e^x$.

$$D = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x & e^x \end{vmatrix} = xe^{2x} - (xe^{2x} + e^{2x}) = -e^{2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ e^x \ln(x) & e^x \end{vmatrix}}{-e^{2x}} = \frac{-\ln(x)e^{2x}}{-e^{2x}} = \ln(x)$$

$$v_2' = \frac{\begin{vmatrix} xe^x & 0 \\ xe^x + e^x & e^x \ln(x) \end{vmatrix}}{-e^{2x}} = \frac{x\ln(x)e^{2x}}{-e^{2x}} = -x\ln(x)$$

$$v_1 = \int \ln(x)dx = x\ln(x) - x$$

$$v_2 = -\int x\ln(x)dx$$

$$u = \ln(x) \quad \Rightarrow \quad du = \frac{dx}{x}, \qquad dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

$$v_2 = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x^2}{2} \times \frac{1}{x} dx\right) = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x}{2} dx\right)$$

$$= -\left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) = \frac{x^2}{4} - \frac{x^2}{2}\ln(x)$$

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The particular solution is

$$y_p = v_1u_1 + v_2u_2 = (x\ln(x) - x)xe^x + \left(\frac{x^2}{4} - \frac{x^2}{2}\ln(x)\right)e^x$$

$$= x^2e^x\ln(x) - x^2e^x + \frac{x^2}{4}e^x - \frac{x^2}{2}e^x\ln(x)$$

$$= \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

The complete solution is

$$y = y_h + y_p = C_1xe^x + C_2e^x + \frac{x^2}{2}e^x\ln(x) - \frac{3x^2}{4}e^x$$

Homework (2)

Solve:

1) $y'' - y' - 2y = e^{3x}$

$$ans: y = c_1e^{-x} + c^2e^{2x} + \frac{1}{4}e^{3x}$$

2) $y'' + y = secx$

$$ans: y = c_1cosx + c_2sinx + ln\,|\,cosx\,|\,cosx + xsinx$$