How to Solving The Differential Equation(D.E.) with Laplace transform (L.T)

So, The derivative property of the laplace transform is:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Example

Solve the following differential equation using Laplace transform

(a) $y'' + 2y' + y = t$ with $y(0) = 0$, and $y'(0) = 1$

(b) $y' + y = 4$ with $y(0) = 0$

(c) $y' + y = \sin(t)$ with $y(0) = 1$

$y'' + 9y = 10\, e^{-t}$	$y(0) = 0, \quad y'(0) = 0$	***Ans:***

Solution

(a) Taking the Laplace transform of the two sides, we get

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{1}{s^2}$$

$$\left(s^2 Y(s) - s(0) - 1\right) + 2\left(sY(s) - 0\right) + Y(s) = \frac{1}{s^2}$$

$$s^2 Y(s) - 1 + 2sY(s) + Y(s) = \frac{1}{s^2}$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1}{s^2} + 1$$

$$Y(s)\left(s^2 + 2s + 1\right) = \frac{1 + s^2}{s^2}$$

$$Y(s)=\frac{1+s^2}{s^2\left(s^2+2s+1\right)}=\frac{1+s^2}{s^2(s+1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{(s+1)}+\frac{D}{(s+1)^2}$$

$$B=\left.\frac{1+s^2}{(s+1)^2}\right|_{s=0}=1$$

$$A=\frac{d}{ds}\left[\frac{1+s^2}{(s+1)^2}\right]_{s=0}=\left.\frac{(s+1)^2(2s)-2\left(1+s^2\right)(s+1)}{(s+1)^4}\right|_{s=0}=\frac{-2}{1}=-2$$

$$D=\left.\frac{1+s^2}{s^2}\right|_{s=-1}=2$$

$$C=\frac{d}{ds}\left[\frac{1+s^2}{s^2}\right]_{s=-1}=\left.\frac{s^2(2s)-\left(1+s^2\right)(2s)}{s^4}\right|_{s=-1}=\frac{-2+4}{1}=2$$

$$Y(s)=\frac{-2}{s}+\frac{1}{s^2}+\frac{2}{s+1}+\frac{2}{(s+1)^2}\quad\Rightarrow\quad y(t)=-2+t+2e^{-t}+2t\cdot e^{-t}$$

(b) Taking the Laplace transform of the two sides, we get

$$sY(s)-y(0)+Y(s)=\frac{4}{s}\quad\Rightarrow\quad sY(s)-0+Y(s)=\frac{4}{s}$$

$$Y(s)(s+1)=\frac{4}{s}\quad\Rightarrow\quad Y(s)=\frac{4}{s(s+1)}$$

$$\frac{4}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

$$A=\left.\frac{4}{(s+1)}\right|_{s=0}=4,\quad\text{and}\quad B=\left.\frac{4}{s}\right|_{s=-1}=-4$$

$$Y(s)=\frac{4}{s}-\frac{4}{s+1}\quad\Rightarrow\quad y(t)=4-4e^{-t}$$

(c) Taking the Laplace transform of the two sides, we get

$$sY(s) - y(0) + Y(s) = \frac{1}{s^2+1} \quad \Rightarrow \quad Y(s)(s+1) = \frac{1}{s^2+1} + 1$$

$$Y(s)(s+1) = \frac{1+s^2+1}{s^2+1} \quad \Rightarrow \quad Y(s) = \frac{s^2+2}{(s+1)(s^2+1)}$$

$$\frac{s^2+2}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A = \left.\frac{s^2+2}{(s^2+1)}\right|_{s=-1} = \frac{(-1)^2+2}{(-1)^2+1} = \frac{3}{2}$$

$$s^2+2 = A(s^2+1) + (Bs+C)(s+1)$$

$$s^2+2 = (A+B)s^2 + (B+C)s + A + C$$

$$1 = A+B \Rightarrow B = 1-\frac{3}{2} = -\frac{1}{2}, \qquad B+C=0 \Rightarrow C = \frac{1}{2}$$

$$Y(s) = \frac{3/2}{s+1} + \frac{(-1/2)s+(1/2)}{s^2+1}$$

$$Y(s) = \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} \quad \Rightarrow \quad y(t) = \frac{3}{2}e^{-t} - \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$$

Example

Solve the following differential equations

(a) $y_1' = -y_2 \qquad y_1(0) = 1$

$y_2' = y_1 \qquad y_2(0) = 0$

(b) $\dfrac{dx}{dt} = 2x - 3y \qquad x(0) = 8$

$\dfrac{dy}{dt} = y - 2x \qquad y(0) = 3$

Solution

(a) Taking the Laplace transform of the two equations, we get

$$sY_1(s) - y_1(0) = -Y_2(s) \quad \Rightarrow \quad sY_1(s) - 1 = -Y_2(s) \quad \Rightarrow \quad sY_1(s) + Y_2(s) = 1$$

$$sY_2(s) - y_2(0) = Y_1(s) \quad \Rightarrow \quad sY_2(s) - 0 = Y_1(s) \quad \Rightarrow \quad -Y_1(s) + sY_2(s) = 0$$

$$\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Y_1(s) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & s \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{s}{s^2+1} \quad \Rightarrow \quad y_1(t) = \cos(t)$$

$$Y_2(s) = \frac{\begin{vmatrix} s & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} s & 1 \\ -1 & s \end{vmatrix}} = \frac{1}{s^2+1} \quad \Rightarrow \quad y_2(t) = \sin(t)$$

(b) Taking the Laplace transform of the two equations, we get

$$sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$sX(s) - 8 = 2X(s) - 3Y(s)$$

$$(s-2)X(s) + 3Y(s) = 8 \qquad . \quad . \quad . \quad (1)$$

$$sY(s) - y(0) = Y(s) - 2X(s)$$

$$sY(s) - 3 = Y(s) - 2X(s)$$

$$2X(s) + (s-1)Y(s) = 3 \qquad . \quad . \quad . \quad (2)$$

$$\begin{bmatrix} s-2 & 3 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

$$X(s)=\frac{\begin{vmatrix}8 & 3\\ 3 & s-1\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{8(s-1)-3\times 3}{(s-2)(s-1)-3\times 2}=\frac{8s-8-9}{s^2-3s+2-6}=\frac{8s-17}{s^2-3s-4}$$

$$X(s)=\frac{8s-17}{(s+1)(s-4)}=\frac{A}{s+1}+\frac{B}{s-4}$$

$$A=\left.\frac{8s-17}{s-4}\right|_{s=-1}=\frac{8(-1)-17}{-1-4}=\frac{-25}{-5}=5$$

$$B=\left.\frac{8s-17}{s+1}\right|_{s=4}=\frac{8(4)-17}{4+1}=\frac{15}{5}=3$$

$$X(s)=\frac{5}{s+1}+\frac{3}{s-4}\quad\Rightarrow\quad x(t)=5e^{-t}+3e^{4t}$$

$$Y(s)=\frac{\begin{vmatrix}s-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}=\frac{3(s-2)-8\times 2}{(s-2)(s-1)-3\times 2}=\frac{3s-6-16}{s^2-3s+2-6}=\frac{3s-22}{s^2-3s-4}$$

$$Y(s)=\frac{3s-22}{(s+1)(s-4)}=\frac{C}{s+1}+\frac{D}{s-4}$$

$$C=\left.\frac{3s-22}{s-4}\right|_{s=-1}=\frac{3(-1)-22}{-1-4}=\frac{-25}{-5}=5$$

$$D=\left.\frac{3s-22}{s+1}\right|_{s=4}=\frac{3(4)-22}{4+1}=\frac{-10}{5}=-2$$

$$Y(s)=\frac{5}{s+1}-\frac{2}{s-4}\quad\Rightarrow\quad y(t)=5e^{-t}-2e^{4t}$$

Homework

Solve The following D.E:

1) $4y'' + \pi^2 y = 0$ $\quad y(0) = 2, \quad y'(0) = 0$

Ans: $2cos\frac{\pi}{2}t$

2) $y' + 4y = 0$ $\quad y(0) = 2.8,$

Ans: $2.8e^{-4t}$

3) $y'' - ky' = 0$ $\quad y(0) = 2 \quad y'(0) = k$

Ans: $u(t) + e^{kt}$

4) $y'' - \frac{1}{4}y = 0$ $\quad y(0) = 4, \quad y'(0) = 0$

Ans: $4cosh\frac{1}{2}t$

5) $y'' + y' - 2y = 0$ $\quad y(0) = 2.1, \quad y'(0) = 3.9$

Ans: $\frac{1}{-3}e^{t} - \frac{1}{3}e^{-2t}$

6) $y' + 2y = 17sin(2t)$ $\quad y(0) = 1, \quad y'(0) = -3$

Ans:

7) $y'' + y' - 2y = e^{-t}$ $\quad y(0) = 2, \quad y'(0) = 0$

Ans: $-\frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^{t}$

8) $y_1' = y_2, \quad y_2' = y_1$ $\quad y_1(0) = 8, \quad y_2(0) = 0$

9) $y_1' + y_2 = 2cos(t), \; y_2' + y_1 = 0$

$y_1(0) = 0, \quad y_2(0) = 1$

Ans: $y_1(t)$ =*sint*

10) $\frac{dx}{dt} = 5x + 3y$ $\quad x(0) = 4$

$\frac{dy}{dt} = y - x = 3y$ $\quad y(0) = 8$