Second Order Linear Homogeneous Equations

The linear equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x)$$

if $F(x) = 0$ then it is called ***homogeneous***; otherwise it is called ***non-homogeneous***.

Linear Differential Operator

It is convenient to introduce the symbol D to represent the operation of differentiation with respect to x. That is, we write $Df(x)$ to mean df/dx. Furthermore, we define powers of D to mean taking successive derivatives:

$$D^2 f(x) = D\{Df(x)\} = \frac{d^2 f}{dx^2}, \qquad D^3 f(x) = D\{D^2 f(x)\} = \frac{d^3 f}{dx^3}$$

$$(D^2 + D - 2)f(x) = D^2 f(x) + Df(x) - 2f(x) = \frac{d^2 f}{dx^2} + \frac{df}{dx} - 2f(x)$$

The Characteristic Equation

The linear second order equation with constant real-number coefficients is

$$\frac{d^2 y}{dx^2} + 2a\frac{dy}{dx} + by = 0$$

or, in operator notation

$$(D^2 + 2aD + b)y = 0$$

$$(D - r_1)(D - r_2)y = 0$$

Solution of $\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0$	
Roots r_1 ***&*** r_2	***Solution***
Real and unequal	$y=C_1e^{r_1x}+C_2e^{r_2x}$
Real and equal	$y=(C_1x+C_2)e^{r_2x}$
Complex conjugate, $\alpha \pm j\beta$	$y=e^{\alpha x}(C_1\cos\beta x \; + \; C_2\sin\beta x)$

Example

Solve the following differential equations:

(a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$, (b) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$

(c) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$, (d) $\frac{d^2y}{dx^2}+4y=0$

Solution

(a)

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

The characteristic equation is

$$D^2+D-2=0$$

$$(D-1)(D+2)=0 \quad \Rightarrow \quad r_1=1 \quad \text{and} \quad r_2=-2$$

The solution is

$$y=C_1e^{x}+C_2e^{-2x}$$

(b) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

The characteristic equation is

$$D^2+4D+4=0$$

$$(D+2)^2=0 \qquad \Rightarrow \qquad r_1=r_2=-2$$

The solution is

$$y=(C_1x+C_2)e^{-2x}$$

(c) $$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+6y=0$$

The characteristic equation is

$$D^2+4D+6=0$$

$$r_{1,2}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

$$r_{1,2}=\frac{-4\pm\sqrt{(4)^2-4(1)(6)}}{2(1)}=\frac{-4\pm\sqrt{16-24}}{2}$$

$$r_{1,2}=\frac{-4\pm\sqrt{-8}}{2}=\frac{-4\pm j2\sqrt{2}}{2}$$

$$r_{1,2}=-2\pm j\sqrt{2} \quad \Rightarrow \quad r_1=-2+j\sqrt{2} \quad \text{and} \quad r_2=-2-j\sqrt{2}$$

$$\Rightarrow \quad \alpha=-2 \quad \text{and} \quad \beta=\sqrt{2}$$

The solution is

$$y=e^{-2x}(C_1\cos\sqrt{2}x \quad + \quad C_2\sin\sqrt{2}x)$$

(d) $$\frac{d^2y}{dx^2} + 4y = 0$$

The characteristic equation is

$$D^2 + 4 = 0$$

$$(D - j2)(D + j2) = 0 \quad \Rightarrow \quad r_1 = j2 \quad \text{and} \quad r_2 = -j2$$

$$\Rightarrow \quad \alpha = 0 \quad \text{and} \quad \beta = 2$$

The solution is

$$y = C_1 \cos 2x \quad + \quad C_2 \sin 2x$$

Second Order Non-homogeneous Linear Equations

Now, we solve non-homogeneous equations of the form

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$$

The procedure has three basic steps. First, we find the homogeneous solution y_h (h stands for "homogeneous") of the ***reduced equation***

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = 0$$

Second, we find a particular solution y_p of the ***complete*** equation. Finally, we add y_p to y_h to form the general solution of the complete equation. So, the final solution is

$$y = y_h + y_p$$

Variation of Parameters

This method assumes we already know the homogeneous solution

$$y_h = C_1 u_1(x) + C_2 u_2(x)$$

The method consists of replacing the constants C_1 and C_2 by functions $v_1(x)$ and $v_2(x)$ and then requiring that the new expression

$$y_h = v_1 u_1 + v_2 u_2$$

and by solving the following two equations

$$v_1' u_1 + v_2' u_2 = 0$$

$$v_1' u_1' + v_2' u_2' = F(x)$$

for the unknown functions v_1' and v_2' using the following matrix notation

$$\begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ F(x) \end{bmatrix}$$

Finally v_1 and v_2 can be found by integration.

In applying the method of ***variation of parameters*** to find the particular solution, the following steps are taken:

i. Find v_1' and v_2' using the following equations

$$v_1' = \frac{\begin{vmatrix} 0 & u_2 \\ F(x) & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{-u_2 F(x)}{D}, \qquad v_2' = \frac{\begin{vmatrix} u_1 & 0 \\ u_1' & F(x) \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} = \frac{u_1 F(x)}{D}$$

where $$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

ii. Integrate v_1' and v_2' to find v_1 and v_2.

iii. Write the particular solution as

$$y_p = v_1 u_1 + v_2 u_2$$

Example

Solve the equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 6$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

The characteristic equation is $D^2 + 2D - 3 = 0$ and the roots of this equation are $r_1 = -3$ and $r_2 = 1$, so

$$y_h = C_1 e^{-3x} + C_2 e^{x}$$

Then

$$u_1 = e^{-3x},\ u_2 = e^{x}$$

$$D = \begin{vmatrix} e^{-3x} & e^{x} \\ -3e^{-3x} & e^{x} \end{vmatrix} = e^{-2x} + 3e^{-2x} = 4e^{-2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{x} \\ 6 & e^{x} \end{vmatrix}}{4e^{-2x}} = \frac{-6e^{x}}{4e^{-2x}} = -\frac{3}{2}e^{3x}, \qquad v_2' = \frac{\begin{vmatrix} e^{-3x} & 0 \\ -3e^{-3x} & 6 \end{vmatrix}}{4e^{-2x}} = \frac{6e^{-3x}}{4e^{-2x}} = \frac{3}{2}e^{-x}$$

$$v_1 = \int -\frac{3}{2}e^{3x}dx = -\frac{1}{2}e^{3x}, \qquad v_2 = \int \frac{3}{2}e^{-x}dx = -\frac{3}{2}e^{-x}$$

$$y_p = v_1u_1 + v_2u_2 = \left(-\frac{1}{2}e^{3x}\right)e^{-3x} + \left(-\frac{3}{2}e^{-x}\right)e^{x} = -2$$

$$y = y_h + y_p = C_1e^{-3x} + C_2e^{x} - 2$$

Example

Solve the equation $y'' - 2y' + y = e^{x}\ln(x)$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$D^2 - 2D + 1 = 0$$

$$(D-1)^2 = 0$$

The roots are $\quad r_1 = r_2 = 1$

The solution is $\quad y_h = (C_1 x + C_2)e^x$

$$y_h = C_1 x e^x + C_2 e^x$$

From that we have $u_1(x) = xe^x$, and $u_2(x) = e^x$.

$$D = \begin{vmatrix} xe^x & e^x \\ xe^x + e^x & e^x \end{vmatrix} = xe^{2x} - (xe^{2x} + e^{2x}) = -e^{2x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ e^x \ln(x) & e^x \end{vmatrix}}{-e^{2x}} = \frac{-\ln(x)e^{2x}}{-e^{2x}} = \ln(x)$$

$$v_2' = \frac{\begin{vmatrix} xe^x & 0 \\ xe^x + e^x & e^x \ln(x) \end{vmatrix}}{-e^{2x}} = \frac{x\ln(x)e^{2x}}{-e^{2x}} = -x\ln(x)$$

$$v_1 = \int \ln(x)dx = x\ln(x) - x$$

$$v_2 = -\int x\ln(x)dx$$

$$u = \ln(x) \quad \Rightarrow \quad du = \frac{dx}{x}, \qquad dv = xdx \quad \Rightarrow \quad v = \frac{x^2}{2}$$

$$v_2 = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x^2}{2} \times \frac{1}{x} dx\right) = -\left(\frac{x^2}{2}\ln(x) - \int \frac{x}{2} dx\right)$$

$$= -\left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) = \frac{x^2}{4} - \frac{x^2}{2}\ln(x)$$

The particular solution is

$$y_p = v_1 u_1 + v_2 u_2 = (x\ln(x) - x)xe^x + \left(\frac{x^2}{4} - \frac{x^2}{2}\ln(x)\right)e^x$$

$$= x^2 e^x \ln(x) - x^2 e^x + \frac{x^2}{4}e^x - \frac{x^2}{2}e^x \ln(x)$$

$$= \frac{x^2}{2}e^x \ln(x) - \frac{3x^2}{4}e^x$$

The complete solution is

$$y = y_h + y_p = C_1 xe^x + C_2 e^x + \frac{x^2}{2}e^x \ln(x) - \frac{3x^2}{4}e^x$$

Undetermined Coefficients

This method gives us the particular solution for selected equations.

The Method of Undetermined Coefficients for Selected Equations of the Form $\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = F(x)$	
If $F(x)$ ***has a term of***	***The expression for*** y_p
A ***(Constant)***	C ***(Another Constant)***
e^{rx}	Ae^{rx}
$\sin(kx)$, $\cos(kx)$	$B\cos(kx) + C\sin(kx)$
$ax^2 + bx + c$	$Dx^2 + Ex + F$

Example

Solve the equation $y'' + 3y = e^x$

Solution

The homogeneous solution y_h can be found using the reduced equation

$$y'' + 3y = 0$$

The characteristic equation is

$$D^2 + 3 = 0$$

The roots are $r_1 = j\sqrt{3}$, and $r_2 = -j\sqrt{3} \Rightarrow \alpha = 0$ and $\beta = \sqrt{3}$

So, $$y_h = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)$$

Since $F(x) = e^x$ then let $y_p = Ae^x \Rightarrow y_p' = Ae^x \Rightarrow y_p'' = Ae^x$

Substituting into the differential equation $y'' + 3y = e^x$ we get

$$Ae^x + 3Ae^x = e^x \Rightarrow A + 3A = 1 \Rightarrow A = \frac{1}{4}$$

So, $$y_p = \frac{1}{4}e^x$$

And the complete solution is

$$y = C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x) + \frac{1}{4}e^x$$

Important Note

The expression used for y_p should not have any term similar to the terms of the homogeneous solution. Otherwise, multiply the term that is similar to the homogeneous solution repeatedly by x until it becomes different.

Example

Solve the equation

(a) $y'' - 6y' + 9y = e^{3x}$,

(b) $y'' - y' = 5e^{x} - \sin(2x)$

(c) $y'' - y' - 2y = 4x^3$

Solution

(a) The homogeneous solution y_h can be found using the reduced equation

$$y'' - 6y' + 9y = 0$$

The characteristic equation is

$$D^2 - 6D + 9 = 0$$

$$(D-3)^2 = 0$$

The roots are $r_1 = r_2 = 3$

$$y_h = (C_1x + C_2)e^{3x}$$

Since $F(x) = e^{3x}$ then let $y_p = Ae^{3x}$. But, Ae^{3x} is similar to the second term of the homogeneous solution so, let $y_p = Axe^{3x}$. Again Axe^{3x} is also similar to the first term of the homogeneous solution. Finally, let

$$y_p = Ax^2e^{3x} \quad \Rightarrow \quad y_p' = 3Ax^2e^{3x} + 2Axe^{3x}$$

$$y_p'' = \left(9Ax^2e^{3x} + 6Axe^{3x}\right) + \left(6Axe^{3x} + 2Ae^{3x}\right)$$
$$= 9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}$$

Substituting into the differential equation $y'' - 6y' + 9y = e^{3x}$ we get

$$\left(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}\right) - 6\left(3Ax^2e^{3x} + 2Axe^{3x}\right) + 9Ax^2e^{3x} = e^{3x}$$
$$2Ae^{3x} = e^{3x}$$
$$\Rightarrow \qquad 2A = 1$$
$$\Rightarrow \qquad A = \frac{1}{2}$$

So, $$y_p = \frac{1}{2}x^2e^{3x}$$

The general solution is $$y = (C_1x + C_2)e^{3x} + \frac{1}{2}x^2e^{3x}$$

b) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' = 0$$

The characteristic equation is

$$D^2 - D = 0$$
$$D(D-1) = 0$$

The roots are $r_1 = 1$, and $r_2 = 0$

$$y_h = C_1e^x + C_2$$

Since $F(x) = 5e^x - \sin(2x)$ then let $y_p = Ae^x + B\cos(2x) + C\sin(2x)$. But, Ae^x is similar to the first term of the homogeneous solution so, let

$$y_p = Axe^x + B\cos(2x) + C\sin(2x)$$

$$y_p' = Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)$$

$$y_p'' = Axe^x + Ae^x + Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

$$= Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)$$

Substituting into the differential equation $y'' - y' = 5e^x - \sin(2x)$ we get

$$\left(Axe^x + 2Ae^x - 4B\cos(2x) - 4C\sin(2x)\right)$$
$$-\left(Axe^x + Ae^x - 2B\sin(2x) + 2C\cos(2x)\right) = 5e^x - \sin(2x)$$

$$Ae^x - (4B + 2C)\cos(2x) + (2B - 4C)\sin(2x) = 5e^x - \sin(2x)$$

$\Rightarrow$ $$A = 5,\quad (4B + 2C) = 0,\quad (2B - 4C) = -1$$

or $$A = 5,\qquad B = -\frac{1}{10},\qquad C = \frac{1}{5}$$

So, $$y_p = 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

The general solution is

$$y = y_h + y_p = C_1e^x + C_2 + 5xe^x - \frac{1}{10}\cos(2x) + \frac{1}{5}\sin(2x)$$

(c) The homogeneous solution y_h can be found using the reduced equation

$$y'' - y' - 2y = 0$$

The characteristic equation is

$$D^2 - D - 2 = 0$$

$$(D-2)(D+1)=0$$

The roots are $r_1 = 2$, and $r_2 = -1$

$$y_h = C_1e^{2x} + C_2e^{-x}$$

Since $F(x) = 4x^3$ then let

$$y_p = Ax^3 + Bx^2 + Cx + D \quad \Rightarrow \quad y_p' = 3Ax^2 + 2Bx + C$$

$$y_p'' = 6Ax + 2B$$

Substituting into the differential equation $y'' - y' - 2y = 4x^3$ we get

$$6Ax + 2B - \left(3Ax^2 + 2Bx + C\right) - 2\left(Ax^3 + Bx^2 + Cx + D\right) = 4x^3$$

$$-2Ax^3 - (3A + 2B)x^2 + (6A - 2B - 2C)x + (2B - C - 2D) = 4x^3$$

$\Rightarrow \quad A = -2$

$$3A + 2B = 0 \quad \Rightarrow \quad 3(-2) + 2B = 0 \quad \Rightarrow \quad B = 3$$

$$6A - 2B - 2C = 0 \quad \Rightarrow \quad 6(-2) - 2(3) - 2C = 0 \quad \Rightarrow \quad C = -9$$

$$2B - C - 2D = 0 \quad \Rightarrow \quad 2(3) - (-9) - 2D = 0 \quad \Rightarrow \quad D = \frac{15}{2}$$

So, $y_p = -2x^3 + 3x^2 - 9x + 7.5$

The general solution is

$$y = C_1e^{2x} + C_2e^{-x} - 2x^3 + 3x^2 - 9x + 7.5$$

Example

- $y'' = 9x^2 + 2x - 1$

  $D^2 = 0 \;\Rightarrow\; r_1 = r_2 = 0 \;\Rightarrow\; y_h = C_1 x + C_2$

  $y_p = x^2\left(Ax^2 + Bx + C\right)$

- $y'' - y' = x$

  $D^2 - D = 0$

  $D(D-1) = 0 \;\Rightarrow\; r_1 = 0 \text{ and } r_2 = 1 \;\Rightarrow\; y_h = C_1 + C_2 e^x$

  $y_p = x\left(Ax + B\right)$

- $y'' - 5y = 3e^x - 2x + 1$

  $D^2 - 5 = 0$

  $(D-\sqrt{5})(D+\sqrt{5}) = 0 \;\Rightarrow\; r_1 = \sqrt{5} \text{ and } r_2 = -\sqrt{5}$

  $y_h = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x}$

  $y_p = Ae^x + Bx + C$

- $y'' - 4y' + 3y = e^{3x} + 2$

  $D^2 - 4D + 3 = 0$

  $(D-3)(D-1) = 0 \;\Rightarrow\; r_1 = 3 \text{ and } r_2 = 1 \;\Rightarrow\; y_h = C_1 e^{3x} + C_2 e^x$

  $y_p = Axe^{3x} + B$

- $y'' + y = 6e^x + 6\cos(x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1 \cos(x) + C_2 \sin(x)$

  $y_p = Ae^{3x} + x\big(B\cos(x) + C\sin(x)\big)$

- $y'' - 2y' + y = xe^x$

  $D^2 - 2D + 1 = 0$

  $(D-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Rightarrow y_h = (C_1 x + C_2)e^x$

  $y_p = (Ax + B)(x^2 e^x)$

- $y'' + y = x^2 \sin(2x)$

  $D^2 + 1 = 0 \Rightarrow r_1 = j$ and $r_2 = -j \Rightarrow \alpha = 0,\ \beta = 1$

  $y_h = C_1 \cos(x) + C_2 \sin(x)$

  $y_p = \left(Ax^2 + Bx + C\right)\ \big(\cos(2x) + \sin(2x)\big)$

Notes:

To find the roots of an equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$

- r is a root of $f(x)$ if $f(r) = 0$.
- r is a repeated root of $f(x)$ if $f'(r) = 0$.
- If r is a root then r must be a factor of a_n.
- If r is a root then $f(x)$ is divided by $(x - r)$.