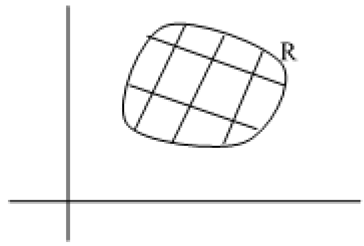
Integral

Multiple Integral

Double Integral Over Nonrectangular Region

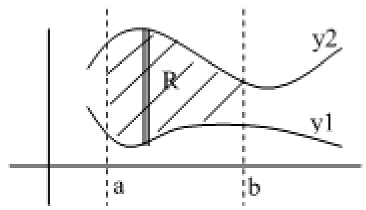
Observation: let R be closed region in the (x, y)- plane. If f is a function of two variables that is define on the region R, then the double integrals on R is written by



$$\lim_{\substack{n\to\infty \\ \Delta A_r\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\Delta A_r = \iint_R f(x,y)\ dA$$

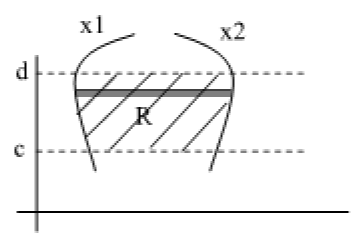
اذا كانت المنحنيات بهذه الصيغة يؤخذ المقطع شاقولي dydx

$$\iint_R f(x,y)\ dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

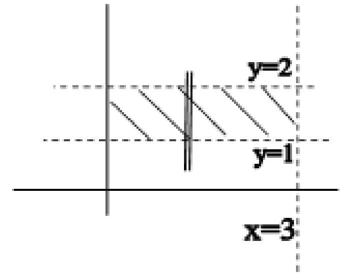


اما اذا كانت المنحنيات بالشكل التالي يؤخذ المقطع افقيا dxdy

$$\iint_R f(x,y)\ dA = \int_c^d \int_{x_1}^{x_2} f(x,y)\,dx\,dy$$



Ex. (1): Evaluate $\int_0^3\int_1^2 (1+8xy)\,dydx$



i) sketch: since dxdy $\Rightarrow$ vertical

$y=1$, $y=2$

ii)

$$\int_0^3\int_1^2 (1+8xy)\,dydx = \int_0^3 \left(y+8x\frac{y^2}{2}\right)\Big|_1^2 dx$$

$$= \int_0^3 \{1+12x\}\,dx$$

$$= \left(x+12\frac{x^2}{2}\right)\Big|_0^3$$

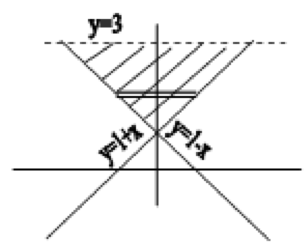
$$= (3+6(9)) - (0) = (3+54) = \mathbf{57}$$

Ex. (2): Evaluate $\iint (2x-y^2)\,dA$ over the triangular R enclosed by

$y=1-x$, $y=1+x$, $y=3$

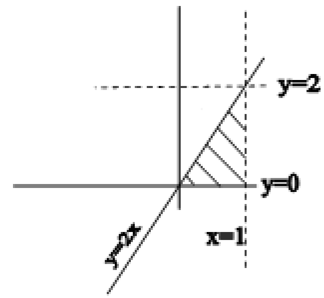
i) sketch:

$y=1-x$,	$y=1+x$
if $x=0 \Rightarrow y=1$,	*if* $x=0 \Rightarrow y=1$
if $y=0 \Rightarrow x=1$,	*if* $y=0 \Rightarrow x=-1$
$\Rightarrow$ (0,1) & (1,0) ,	$\Rightarrow$ (0,1) & (-1,0)



$$\left|\iint_R (2x-y^2)\,dA = \int_1^3\int_{1-y}^{y-1} (2x-y^2)\,dxdy\right| = \int_1^3 (x^2-y^2x)\Big|_{1-y}^{y-1} dy = \left|18-\frac{244}{6}\right|$$

Ex. (3): Evaluate $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy$



Reverse the order of integration

Since *dxdy* horizontal

$x = \frac{y}{2} \quad \Rightarrow \quad y = 2x$

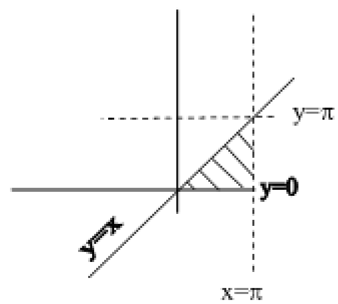
$x = 1$

for y from $0 \to 2$

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} \, dx \, dy = \int_0^1 \int_0^{2x} e^{x^2} \, dy \, dx \quad = \int_0^1 e^{x^2} y \Big|_0^{2x} dx$$

$$= \mathbf{e - 1}$$

Ex. (4): Evaluate $\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy$



From left $x = y$

From right $x = \pi$

value of y , from $0 \Rightarrow x$

reverse the order

$$\Rightarrow \int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy = \int_0^\pi \int_0^x \frac{\sin x}{x} \, dy \, dx$$

$$= \int_0^\pi \frac{\sin x}{x} \cdot y \Big|_0^x dx \quad = \int_0^\pi \frac{\sin x}{x} \cdot x \, dx$$

$$= \mathbf{2}$$

Ex. (5):

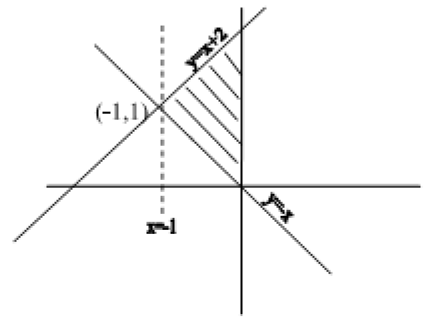
$$\int_0^2 \int_x^2 2y^2 \sin xy \, dy dx = \int_0^2 \int_0^y 2y^2 \sin xy \, dx dy$$

$$= \int_0^2 \left[-2y \cos xy\right]_0^y dy = \int_0^2 \left[-2y \cos y^2 + 2y\right] dy$$

$$= \left[-\sin y^2 + y^2\right]_0^2 = \mathbf{4 - \sin 4}$$

Ex. (6): Write an equivalent double of integration reversed $\int_{-1}^{0}\int_{-x}^{x+2} (x^2+y^2)dy\,dx$

$$\int_{-1}^{0}\int_{-x}^{x+2} (x^2+y^2)dy\,dx$$
$$= \int_{0}^{1}\int_{-y}^{0}(\mathbf{x}^2+\mathbf{y}^2)\mathbf{dxdy} + \int_{1}^{2}\int_{y-2}^{0}(\mathbf{x}^2+\mathbf{y}^2)\mathbf{dxdy}$$



Ex. (7): Draw the region bounded by $y=e^x$, $y=\sin x$, $x=\pi$, $x=-\pi$ and evaluate its area.

$$A=\int_{-\pi}^{\pi}\int_{\sin x}^{e^x} dydx$$
$$=\int_{-\pi}^{\pi} y\Big|_{\sin x}^{e^x}$$
$$=e^{\pi}-e^{-\pi}$$

قوانين و ملاحظات مهمة مع بعض الامثلة حول التكامل

Multiple Integrals

Double Integral over Rectangular Region

If $f(x,y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

Double Integral over Nonrectangular Region

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

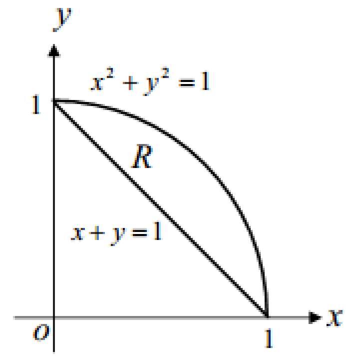
2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 are continuous on $[c,d]$, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

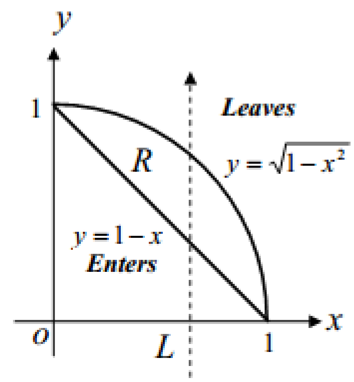
Finding Limits of Integration

To evaluate $\iint_R f(x,y)dA$ and if we integrate first with respect to y and then with respect to x, do the following:

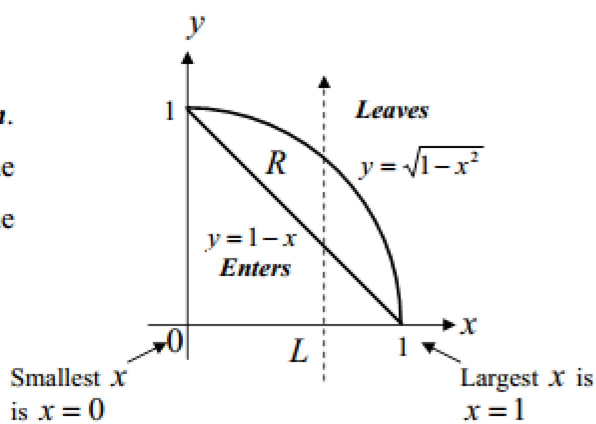
1) ***Sketch***. Sketch the region of integration and label the bounding curves.



2) ***Find the y-limits of integration***. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x.

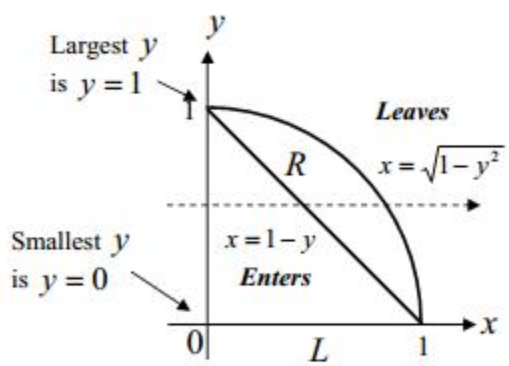


3) ***Find the x-limits of integration***. Choose x-limits that include all the vertical lines through R. The integral becomes



$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in steps 2 and 3. the integral is



$$\iint_R f(x,y)\,dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

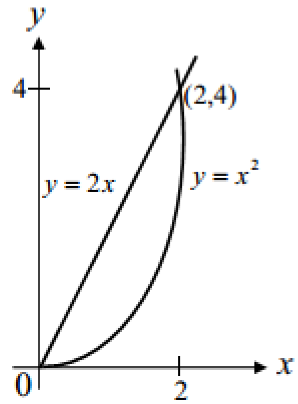
Example

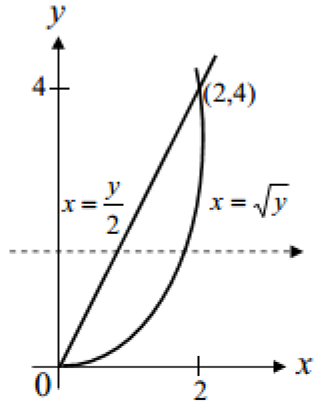
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

and write the equivalent integral with the order of integration reversed.

Solution





To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines y must be from 0 to 4.

So,

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

Example

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y$$

Solution

$$V = \int_0^1 \int_0^x (3 - x - y)\,dy\,dx = \int_0^1 \left(3y - xy - \frac{y^2}{2} \right)\Bigg|_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left(\frac{3x^2}{2} - \frac{x^3}{2} \right)\Bigg|_{x=0}^{x=1} = 1$$

When the order of the integration is reversed, the integral of the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)dxdy = \int_0^1 \left(3x - \frac{x^2}{2} - xy \right)\Bigg|_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right)dy = \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right)dy$$

$$= \left(\frac{5}{2}v - 2y^2 + \frac{1}{2}y^3 \right)\Bigg|_{y=0}^{y=1} = 1$$

Example

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dydx = \int_{-1}^{2} y\Big|_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)dx$$

$$= \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right)\Bigg|_{-1}^{2} = \frac{9}{2}$$

On the other hand, reversing the order of integration results in dividing the region into two parts as follows:

$$A = \iint_{R_1} A_1 dxdy + \iint_{R_2} A_2 dxdy$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy + \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy$$

Homework

Calculate $\iint_R \frac{\sin(x)}{x} dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line y=x, and the line x=1.

Ans.: 0.46

Find the area of the region R bounded by y=x and y=x^2 in the first quadrant.

Ans.: $\frac{1}{6}$