(1.1.2) Exact ODEs. Integrating Factors:

* Exact Test, Exact solution:

A differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if for some function $f(x,y)$

$$M(x,y)\,dx + N(x,y)\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = df$$

is exact if and only if $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$

* Example: The equation $(x^2+y^2)\,dx + (2xy + \cos(y))\,dy = 0$ is exact because the partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy + \cos(y)) = 2y$$

are equal.

- The equation $(x+3y)\,dx + (x^2+\cos(y))\,dy = 0$ is not exact because the partial derivatives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x+3y) = 3$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2+\cos(y)) = 2x$$

are not equal.

Solution of Exact Differential Equation

1. Match the equation to the form $M(x,y)dx + N(x,y)dy = 0$ to identify M and N.
2. Integrate M (or N) with respect to x (or y), writing the constant of integration as $g(y)$ (or $g(x)$)
3. Differentiate with respect to y (or x) and set the result equal to N (or M) to find $\bar{g}(y)$ (or $\bar{g}(x)$)
4. Integrate to find $g(y)$ or $g(x)$
5. Write the Solution of the exact equation as $f(x,y) = C$.

Example: Solve the differential equation

$$(x^2+y^2)dx + (2xy + \cos(y))\,dy = 0$$

Solution:

Step 1: Match the equation to the form $M(x,y)dx + N(x,y)dy = 0$ to identify M.

$$M(x,y) = x^2 + y^2$$

Step 2: Integrate M with respect to x, writing the constant of integration as $g(y)$.

$$f(x,y) = \int M(x,y)\,dx = \int (x^2+y^2)\,dx$$

$$= \frac{x^3}{3} + xy^2 + g(y)$$

Step 3: Differentiate with respect to y and set the result equal to N ~~to~~ find $g'(y)$.

$$\frac{\partial}{\partial y}\left[\frac{x^3}{3} + xy^2 + g(y)\right] = 2xy + g'(y)$$

$$2xy + g'(y) = 2xy + \cos(y)$$

$$\Rightarrow g'(y) = \cos(y)$$

Step 4: Integrate to find $g(y)$

$$\int g'(y)\,dy = \int \cos(y)\,dy = \sin(y)$$

Step 5: Write the solution of the exact equation as $f(x,y) = c$

$$\frac{x^3}{3} + x y^2 + \sin(y) = c$$

* Reducible to Exact: A Defferential equ. $M(x,y)\,dx + N(x,y)\,dy = 0$ which is not exact can be made exact by Multiplying both sides by a suitable integrating factor (ρ). in other words, the equation

$$\rho M(x,y)\,dx + \rho N(x,y)\,dy = 0$$

is an exact equation for an appropriate choice of (ρ)

* Method to find the integrating factor:

* If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ or constant than

$$f(x) = e^{\int f(x)\, dx}$$

* If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ or constant than

$$f(y) = e^{\int f(y)\, dy}$$

<u>Example:</u> Solve the equation $2y\, dx + x\, dy = 0$

<u>Solution:</u> $M(x,y) = 2y \Rightarrow \dfrac{\partial M}{\partial y} = 2$

$N(x,y) = x \Rightarrow \dfrac{\partial N}{\partial x} = 1$

this equation is not exact:

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2-1}{x} = \frac{1}{x} = f(x)$$

$$\int f(x)\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$\therefore\ \rho(x) = e^{\int f(x)\,dx} = e^{\ln x} = x$$

- Multiplying both sides of the equation by integrating factor $\rho(x) = x$, we get

$$x(2y\,dx + x\,dy) = 0 \Longrightarrow 2xy\,dy + x^2\,dx = 0$$

which exact because $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$, and the solution is :

$$f(x,y) = \int 2xy\,dx = x^2 y + g(y)$$

$$\frac{\partial}{\partial y}\left(x^2 y + g(y)\right) = x^2 + \acute{g}(y)$$

$$x^2 + \acute{g}(y) = x^2 \Rightarrow \acute{g}(y) = 0$$

$$g(y) = \int \acute{g}(y)\,dy = c \Longrightarrow x^2 y = c$$

Applied problems:

* Variable Seperable:

1- Solve: $\boxed{y' + (x+2)y^2 = 0}$

Solution:

$$\frac{dy}{dx} = -(x+2)\,y^2$$

$$\frac{dy}{y^2} = -(x+2)\,dx$$

$$\int \frac{dy}{y^2} = -\int (x+2)\,dx \Longrightarrow \boxed{\frac{-1}{y^{2-1}} = -\frac{x^2}{2} - 2x + c}$$

$$-\frac{1}{y} = -\frac{1}{2}x^2 - 2x + c$$

$$y = \frac{2}{x^2 + 4x + c}$$

② $\acute{y} = (4x^2 + y^2) / (xy)$

ملاحظة: في مثل هذا النوع من المعادلات التفاضلية نحتاج إلى تحويلها إلى صيغة separable ونعوض ب $\boxed{u = \frac{y}{x}}$

Solution:

$\acute{y}\ (4x^2 + y^2) / (xy)$

نشتق

$\boxed{u = \frac{y}{x}}$, then $y = xu$, $\acute{y} = u + x\acute{u}$ — ①

$\acute{y} = \frac{4x^2}{\cancel{x}y} + \frac{y^2}{x\cancel{y}} = \frac{4x}{y} + \frac{y}{x} = \frac{4}{u} + u$ — ②

$\acute{y} = \cancel{u} + x\acute{u} = \frac{4}{u} + \cancel{u}$

$x\acute{u} = \frac{4}{u}$

$x\frac{du}{dx} = \frac{4}{u}$

$u\,du = \frac{4}{x}dx$

$\int u\,du = 4\int \frac{1}{x}dx$

$\frac{u^2}{2} = 4\ln|x| + C$

$u^2 = 8\ln|x| + C$

$\because u = \frac{y}{x}$

$\therefore u^2 = \frac{y^2}{x^2}$

نعوض في

$y^2 = x^2(8\ln|x| + c)$

⑬

③ Find a general solution :

$$x y' = \frac{1}{2} y^2 + y \qquad \text{H.W}$$

answer $\boxed{y = x u = \frac{2x}{c - x}}$

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* Exact :

④ $(x - y)(dx - dy) = 0$

Solution :

$(x - y)\,dx + (y - x)\,dy = 0$. Exact ; the test gives (-1) on both sides

$$f(x,y) = \int M(x,y)\,dx = \int (x - y)\,dx = \frac{x^2}{2} - xy + g(y)$$

$$\frac{\partial}{\partial y}\left[\frac{x^2}{2} - xy + g(y)\right] = -x + g'(y)$$

$$-x + g'(y) = y - x \qquad \text{]exact}$$

$$\therefore g'(y) = y \quad \boxed{g(y) = \int y\,dy = \frac{y^2}{2}}$$

نعوض في

$$\frac{x^2}{2} - xy + \frac{y^2}{2} = \frac{1}{2}(x - y)^2 = C$$

(5) Find the general Solution: $\boxed{\text{H.w}}$

$$(e^{y} - y e^{x})\,dx + (x e^{y} - e^{x})\,dy$$

$$\boxed{\begin{array}{l}\text{answer:}\\ x e^{y} - y e^{x} = C\end{array}}$$

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(6) $$\underbrace{\left(2x + \frac{1}{y} - \frac{y}{x^2}\right)}_{M} dx + \underbrace{\left(2y + \frac{1}{x} - \frac{x}{y^2}\right)}_{N} dy = 0$$

Solution: Exact; $-\frac{1}{x^2} - \frac{1}{y^2}$ on both sides of equ.

integrate M with respect to x

$$f(x,y) = \int M(x,y)\,dx = x^2 + \frac{x}{y} + \frac{y}{x} + g(y)$$

Differentiate this with respect to y and equate the result to N

$$\frac{\partial}{\partial y}\left(x^2 + \frac{x}{y} + \frac{y}{x} + g(y)\right) = -\frac{x}{y^2} + \frac{1}{x} + g'(y) = N$$

$$\therefore g'(y) = 2y \Rightarrow g(y) = \int 2y\,dy = y^2$$

Answer:

$$x^2 + \frac{x}{y} + \frac{y}{x} + y^2 = C$$