Digital Signal Processing

Course Instructor
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Lecture No. 2

Third Class
Department of Computer and Software Engineering

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Lecture Outline

- **Classification of Signals**
- **Basic Types of Digital Signals:**
  1) **Unit Step**
  2) **Impulse**
  3) **Ramp**
  4) **Exponential**
  5) **Cosine**
- **Classification of DSP Systems:**
  1) **Causality**
  2) **linearity**
  3) **Time Invariant**
  4) **Stability**
- **Characterization of Digital Filters:**

  (1) Recursive **(2) Non-Recursive**

Classification of Signals

- **Multichannel and Multidimensional Signals**

$$s_1(t) = A \sin 3\pi t$$

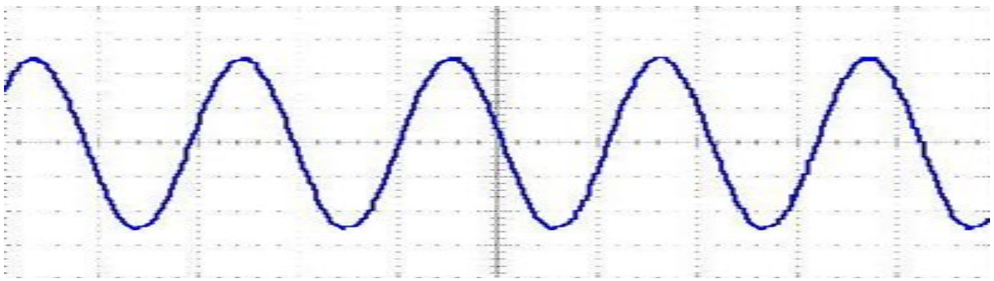
$$s_2(t) = Ae^{j3\pi t} = A \cos 3\pi t + jA \sin 3\pi t$$

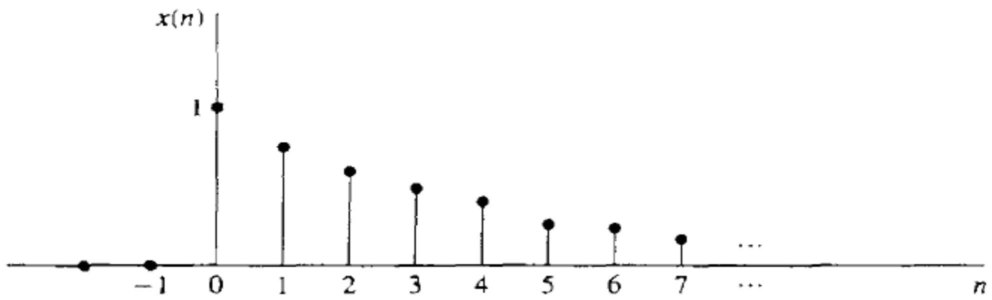
$$\mathbf{S}_3(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}$$

$$\mathbf{I}(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix}$$

Classification of Signals

- **Continuous-Time and Discrete-Time Signals**



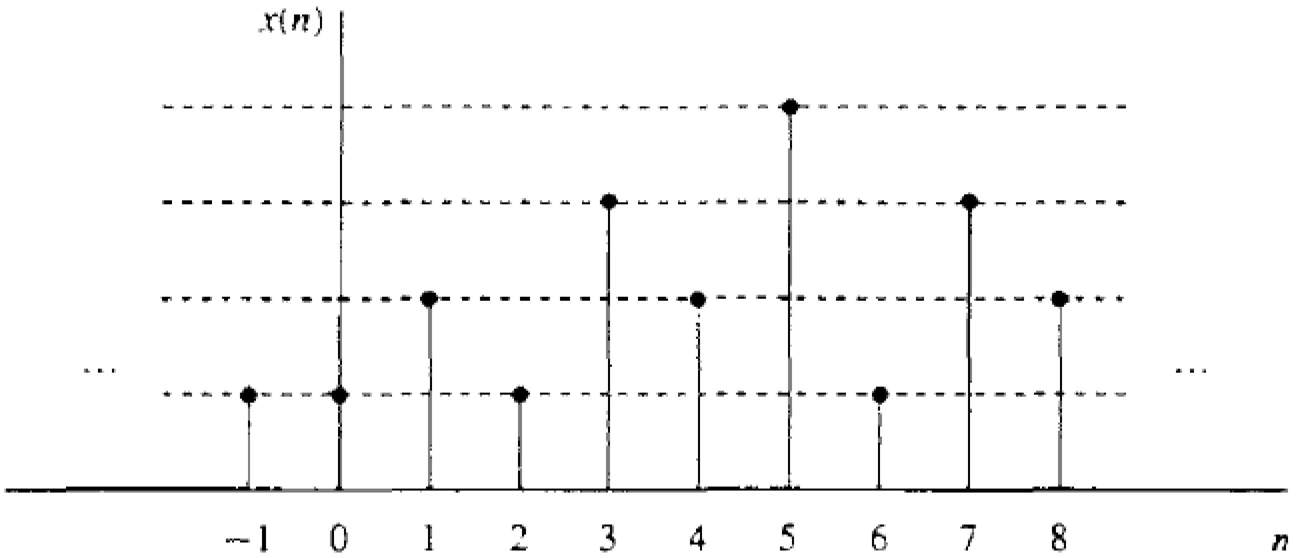


$$x(t) = 0.8^t,\ t \geq 0 \text{ and } x(t) = 0,\ t < 0$$

$$x(n) = \begin{cases} 0.8^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Classification of Signals

- **Continuous-Valued and Discrete-Valued Signals**



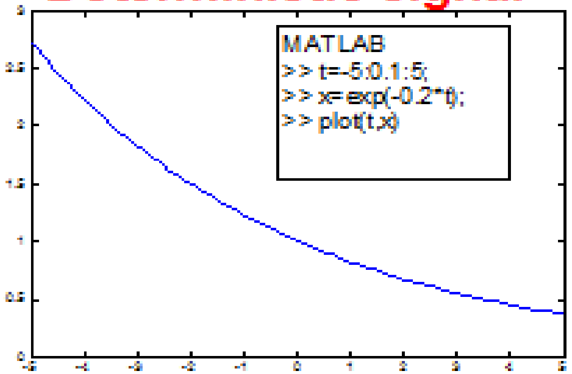
Classification of Signals

- **Deterministic and Random Signals**

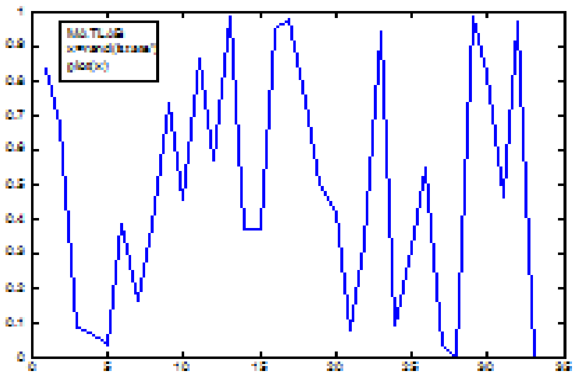
Deterministic Vs Random

- A **deterministic** signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression. The past, present and future of a deterministic signal are known with certainty. Because of this the future values of the signal can be calculated from past values with complete confidence.
  - Example: $x(t) = e^{-2t}$ is a deterministic signal.
- A **random** or stochastic signal has a lot of uncertainty about its behavior. The future values of a random signal can't be accurately predicted. The random signal can be modeled using statistical information about the signal.
  - Examples: some common examples of random signals are speech and music.

Deterministic signal

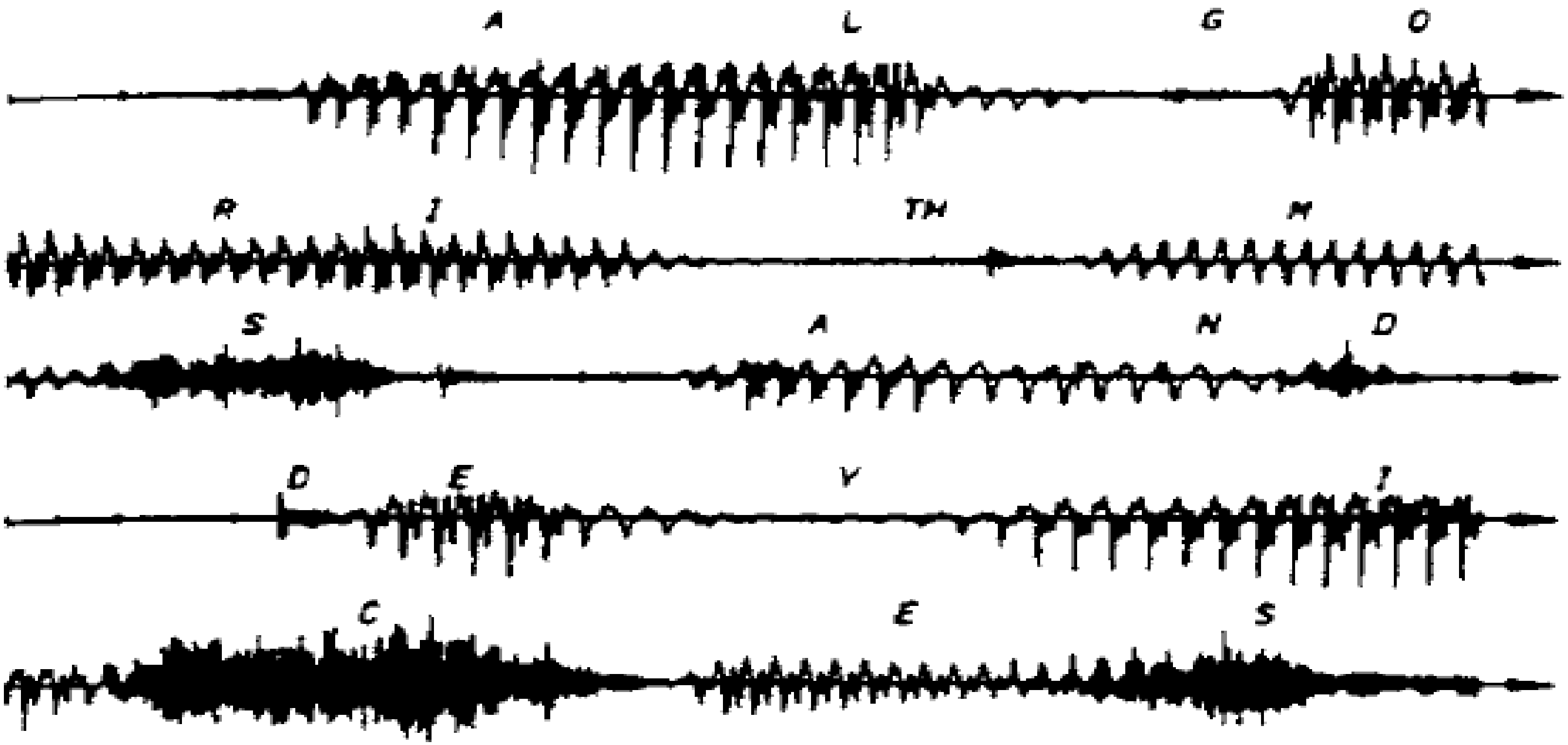


Random signal



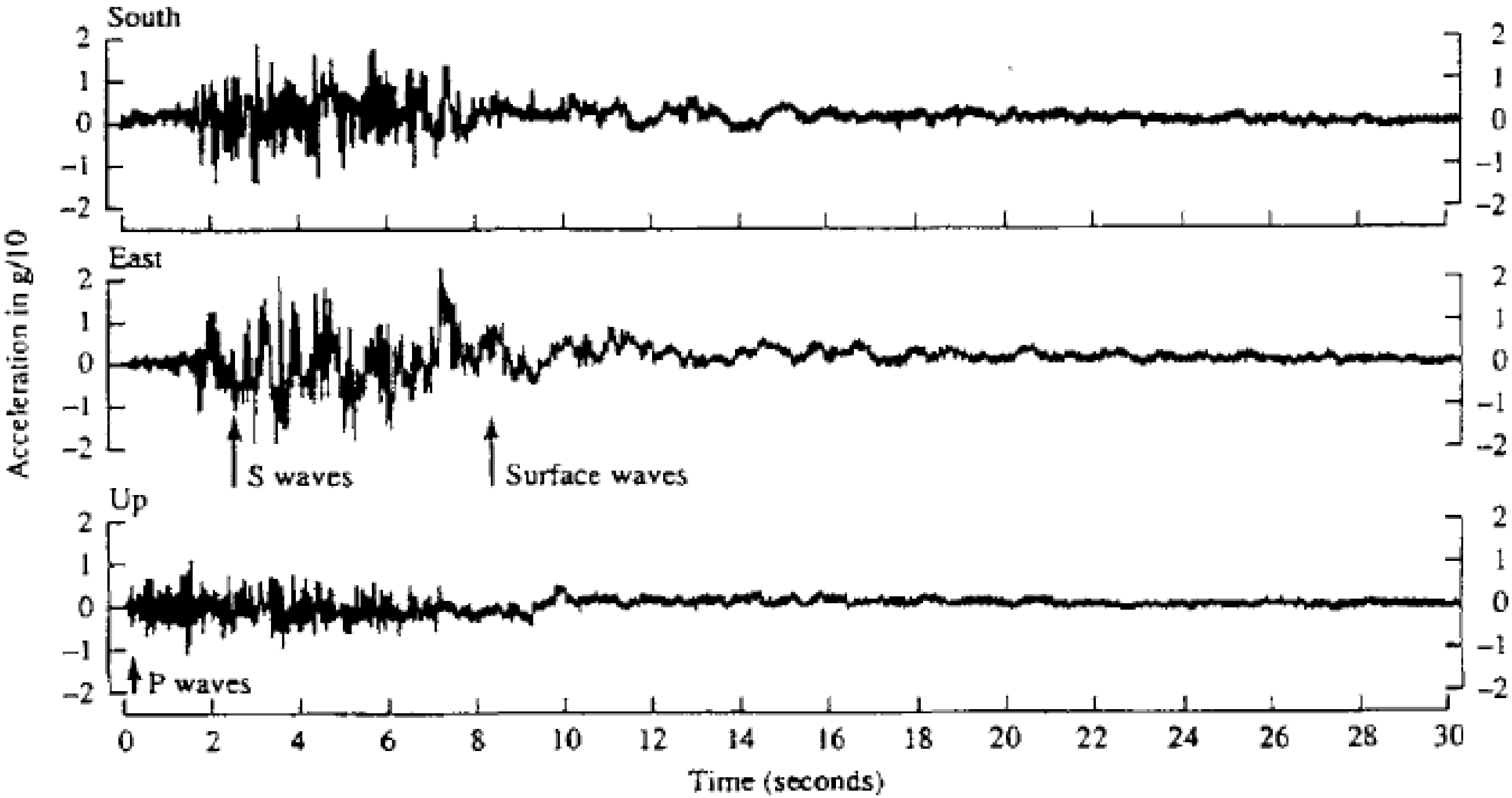
Classification of Signals

- **Deterministic and Random Signals**



Classification of Signals

- **Deterministic and Random Signals**



Classification of Signals

- **Peroidic and Aperoidic Signals**

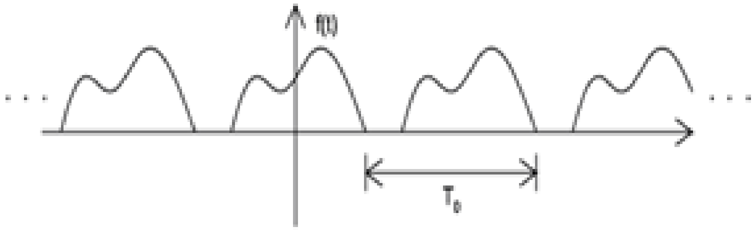
Periodic VS Aperiodic

- Periodic signals repeat with some period T, while aperiodic, or nonperiodic, signals do not. We can define a periodic function through the following mathematical expression, where t can be any number and T is a positive constant

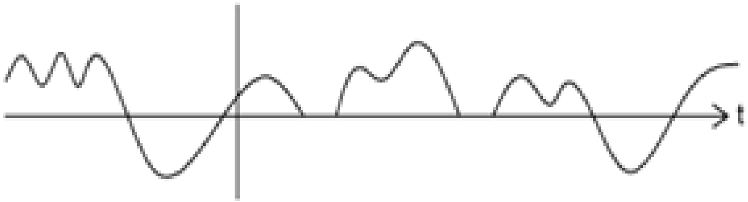
$$x(t) = x(t+T)$$

- The fundamental period of our function, x(t), is the smallest value of T that allows above Equation to be true.

Periodic signals



Aperiodic signals



Classification of Signals

- **Peroidic and Aperoidic Signals**

Defining Periodicity of a discrete-time signal:

- For any *continuous-time* sinusoidal function, $x(t) = A\cos(\Omega_0 t + \theta)$ then it is always periodic with period $T = 2\pi / \Omega_0$.

Example 1: Show that x(t)=x(t+T)= $e^{j\Omega_0 t}$

Solution 1: x(t+T)$e^{j\Omega_0(t+T)} = e^{j\Omega_0 t}.e^{j\Omega_0 T} = e^{j\Omega_0 t}.e^{j\Omega_0(\frac{2\pi}{\Omega_0})} = e^{j\Omega_0 t}.e^{j2\pi}$

Recall that $e^{j2\pi} = \cos(2\pi) + j\sin(2\pi) = 1 + 0 = 1$

Hence, $= e^{j\Omega_0 t} = x(t)$ Proved.

- For a *discrete-time* sinusoid, it may or may not be periodic!
- *So how can we say if a discrete function is periodic or not????*

Classification of Signals

- **Peroidic and Aperoidic Signals**

  - To decide if a discrete function is periodic or not, lets assume, $x(n) = \cos(n\omega_0 + \theta)$ is a periodic signal such that , $x(n) = x(n+N)$ then:

    $$\cos(n\omega_0 + \theta) = \cos([n+N]\omega_0 + \theta) = \cos(n\omega_0 + N\omega_0 + \theta) = \cos(n\omega_0 + \theta + N\omega_0)$$

  - According to our assumption $x(n)$ is a periodic signal, therefore, $N\omega_0$ must be equal to the integer multiple of 2π, thus:

  - $N\omega_0 = l2\pi$ where l is the integer > 0.

Therefore, $\omega_0 = \frac{l}{N} 2\pi$

- So for $x(n) = \cos(n\omega_0 + \theta)$ to be periodic, ω_0 must be a rational multiple of 2π
- The periodicity of $x(n)$ is N, where $\omega_0 = \frac{l}{N} 2\pi$, and l and N are the smallest possible integers.

Continuous-time Sinusoids

To find the period $T > 0$ of a general continuous-time sinusoid $x(t) = A\cos(\omega t + \phi)$:

$$\begin{aligned} x(t) &= x(t+T) \\ A\cos(\omega t + \phi) &= A\cos(\omega(t+T) + \phi) \\ A\cos(\omega t + \phi + 2\pi k) &= A\cos(\omega t + \phi + \omega T) \\ \therefore \quad 2\pi k &= \omega T \\ T &= \frac{2\pi k}{\omega} \end{aligned}$$

where $k \in \mathbb{Z}$. Note: when k is the same sign as ω, $T > 0$.

Therefore, there exists a $T > 0$ such that $x(t) = x(t+T)$ and therefore $x(t)$ is periodic.

Classification of Signals

Discrete Time Sinusoids

Periodicity

Recall if a signal $x(t)$ is periodic, then there exists a $T > 0$ such that

$$x(t) = x(t + T)$$

If no $T > 0$ can be found, then $x(t)$ is non-periodic.

Discrete-time Sinusoids

To find the integer period $N > 0$ (i.e., ($N \in \mathbb{Z}^+$) of a general discrete-time sinusoid $x[n] = A\cos(\Omega n + \phi)$:

$$\begin{aligned} x[n] &= x[n+N] \\ A\cos(\Omega n + \phi) &= A\cos(\Omega(n+N)+\phi) \\ A\cos(\Omega n + \phi + 2\pi k) &= A\cos(\Omega n + \phi + \Omega N) \\ \therefore \quad 2\pi k &= \Omega N \\ N &= \frac{2\pi k}{\Omega} \end{aligned}$$

where $k \in \mathbb{Z}$.

Note: there may not exist a $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\Omega}$ is an integer.

Discrete-time Sinusoids

Example i: $\Omega = \frac{37}{11}\pi$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\frac{37}{11}\pi} = \frac{22}{37}k$$

$$N_0 = \frac{22}{37}k = \boxed{22} \text{ for } k = 37;\ x[n] \text{ is periodic.}$$

Example ii: $\Omega = 2$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{2} = \pi k$$

$$N \in \mathbb{Z}^+ \quad \text{does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is non-periodic.}$$

Example iii: $\Omega = \sqrt{2}\pi$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\sqrt{2}\pi} = \sqrt{2}k$$

$$N \in \mathbb{Z}^+ \quad \text{does not exist for any } k \in \mathbb{Z};\ x[n] \text{ is not periodic.}$$

Discrete-time Sinusoids

$$N = \frac{2\pi k}{\Omega}$$

$$\Omega = \frac{2\pi k}{N} = 2\pi \frac{k}{N} = \pi \cdot \underbrace{\boxed{\frac{2k}{N}}}_{RATIONAL}$$

Therefore, a discrete-time sinusoid is periodic if its radian frequency Ω is a rational multiple of π.

Otherwise, the discrete-time sinusoid is non-periodic.

Classification of Signals

Discrete Time Sinusoids

Example 1: $\Omega = \pi/6 = \pi \cdot \boxed{\frac{1}{6}}$

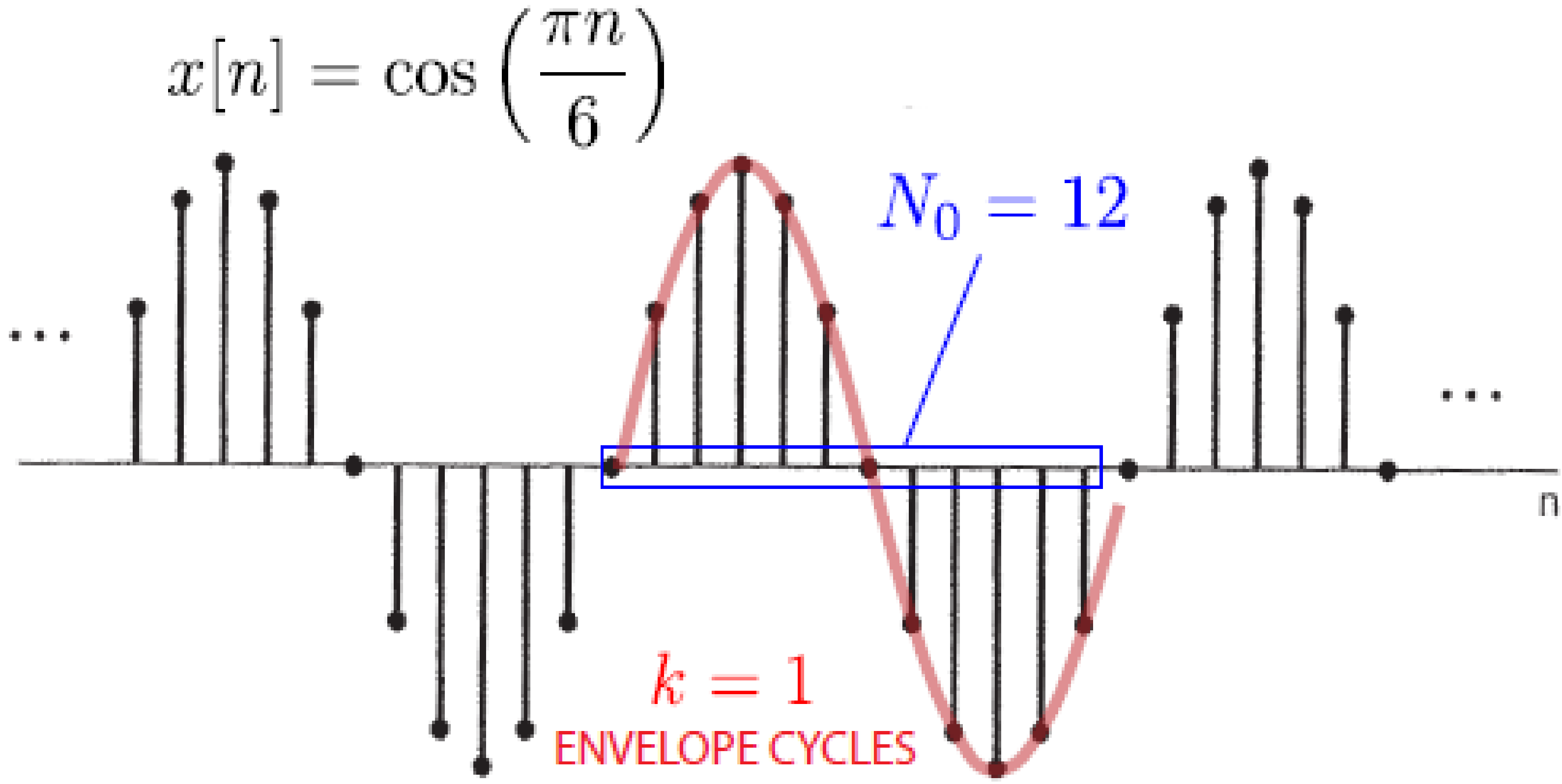
$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$

$$\begin{aligned} N &= \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi \frac{1}{6}} = 12k \\ N_0 &= 12 \quad \text{for } k = 1 \end{aligned}$$

The fundamental period is 12 which corresponds to $k = 1$ envelope cycles.

Classification of Signals

Discrete Time Sinusoids



Classification of Signals

Discrete Time Sinusoids

Example 2: $\Omega = 8\pi/31 = \pi \cdot \boxed{\frac{8}{31}}$

$$x[n] = \cos\left(\frac{8\pi n}{31}\right)$$

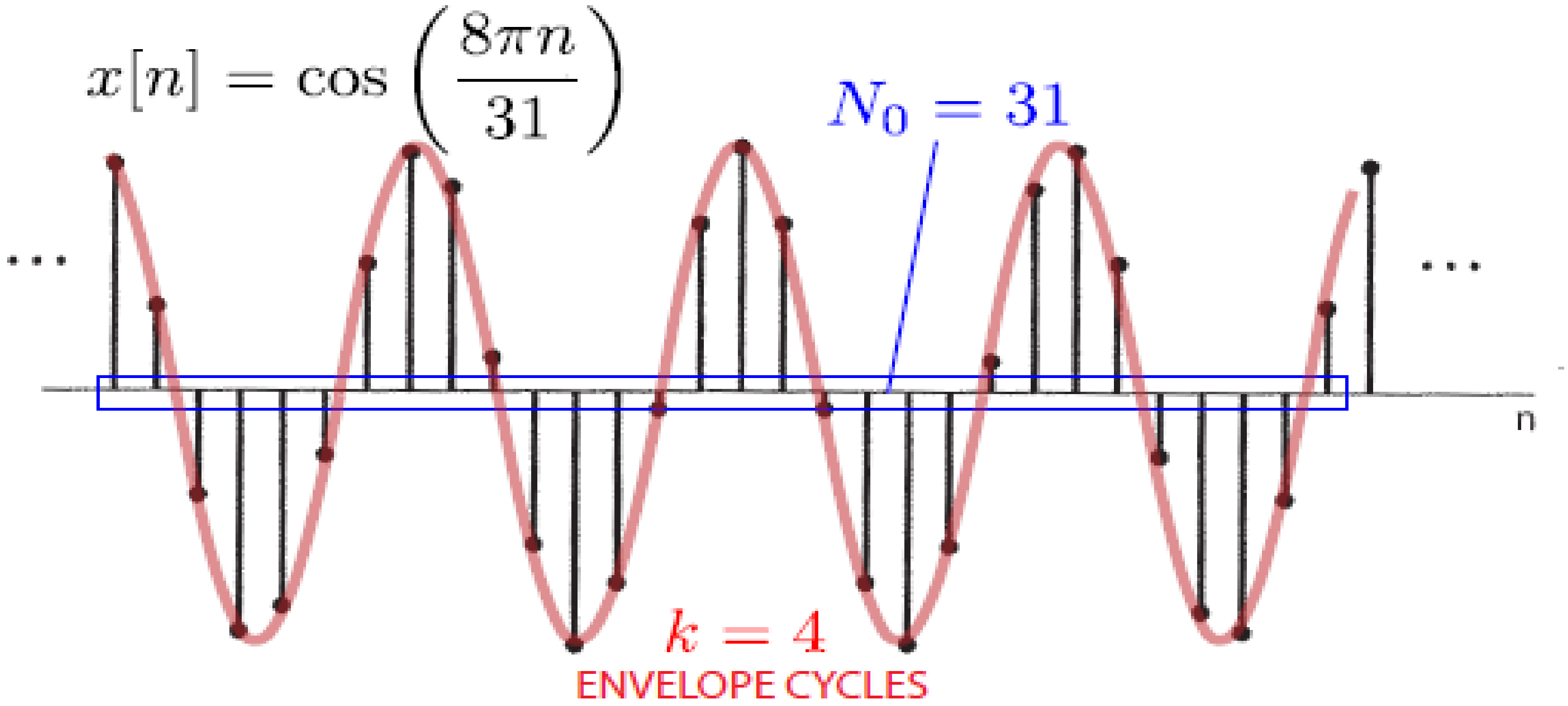
$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\pi \frac{8}{31}} = \frac{31}{4}k$$

$$N_0 = 31 \quad \text{for } k = 4$$

The fundamental period is 31 which corresponds to $k = 4$ envelope cycles.

Classification of Signals

Discrete Time Sinusoids



Example 3: $\Omega = 1/6 = \pi \cdot \boxed{\frac{1}{6\pi}}$

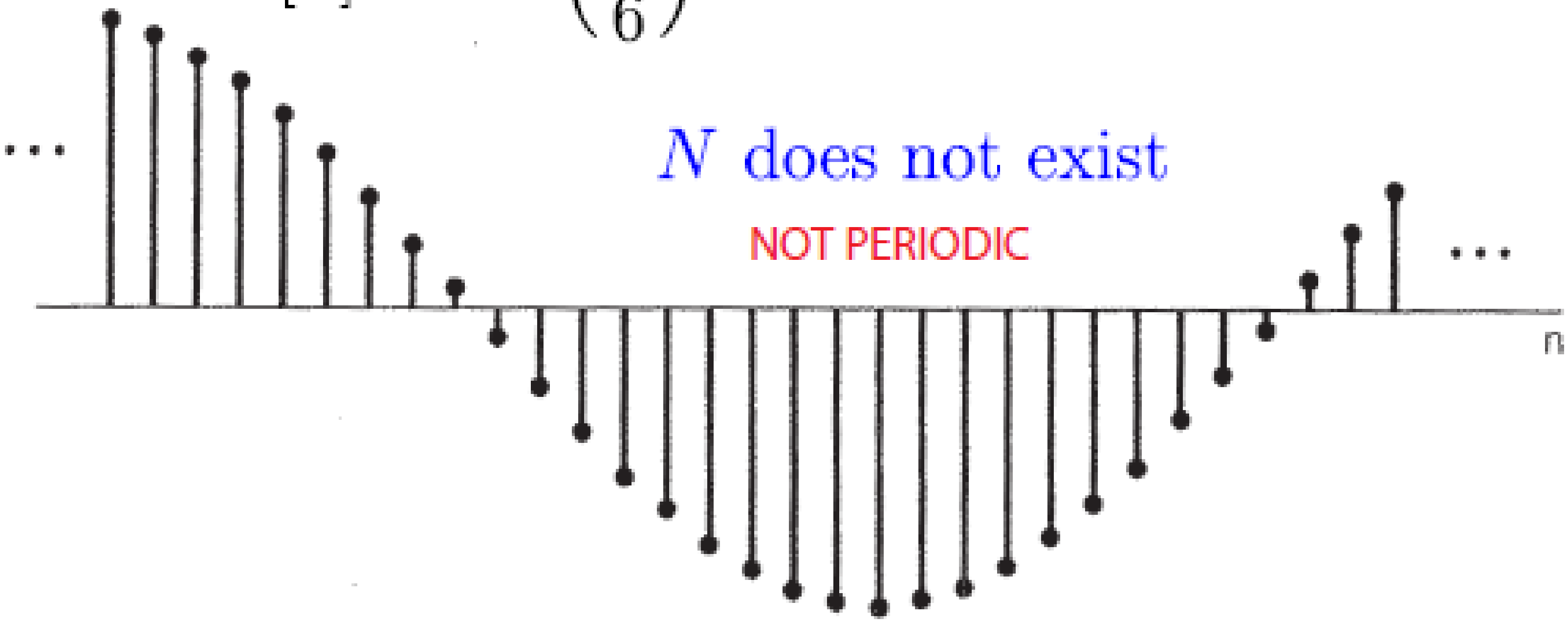
$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$N = \frac{2\pi k}{\Omega} = \frac{2\pi k}{\frac{1}{6}} = 12\pi k$$

$N \in \mathbb{Z}^+$ does not exist for any $k \in \mathbb{Z}$; $x[n]$ is non-periodic.

Discrete Time Sinusoids

$$x[n] = \cos\left(\frac{n}{6}\right)$$



Continuous-Time Sinusoids: Frequency and Rate of Oscillation

$$x(t) = A\cos(\omega t + \phi)$$

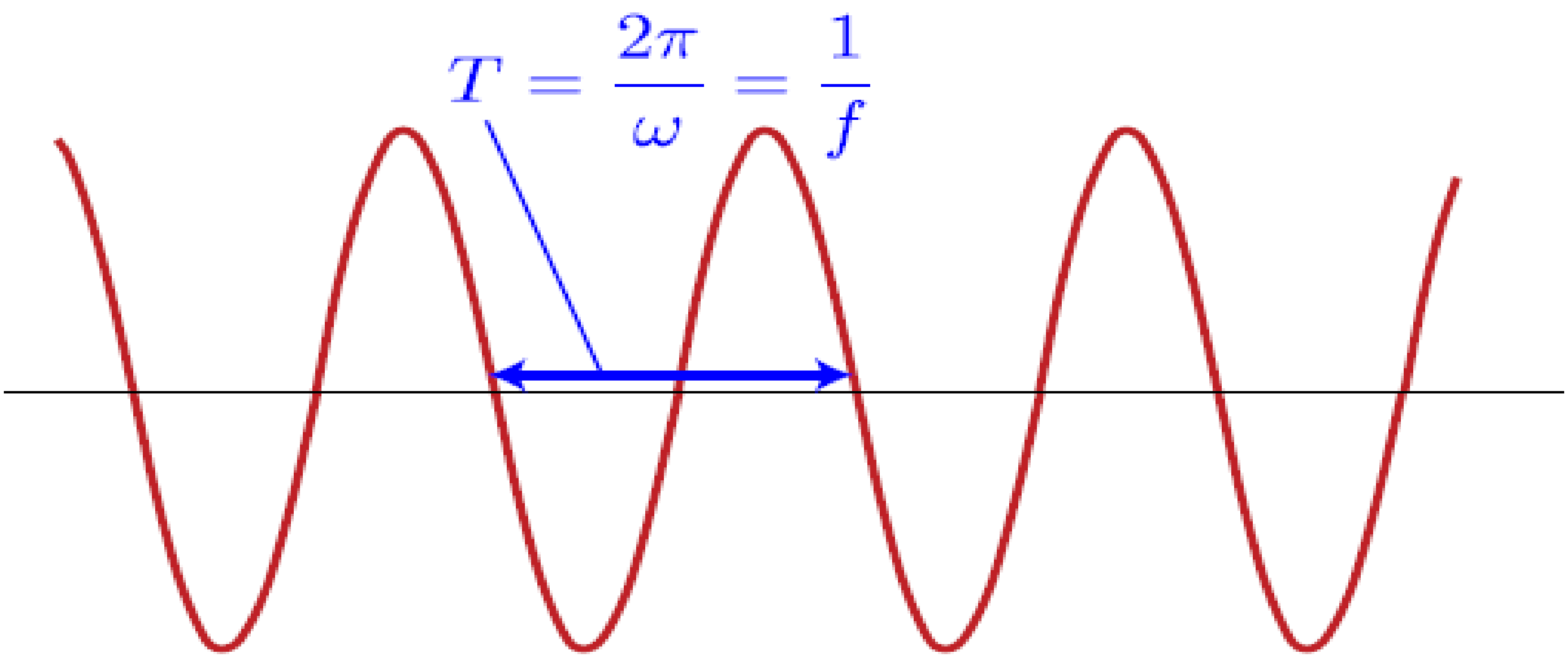
$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Rate of oscillation increases as ω increases (or T decreases).

Classification of Signals

Discrete Time Sinusoids

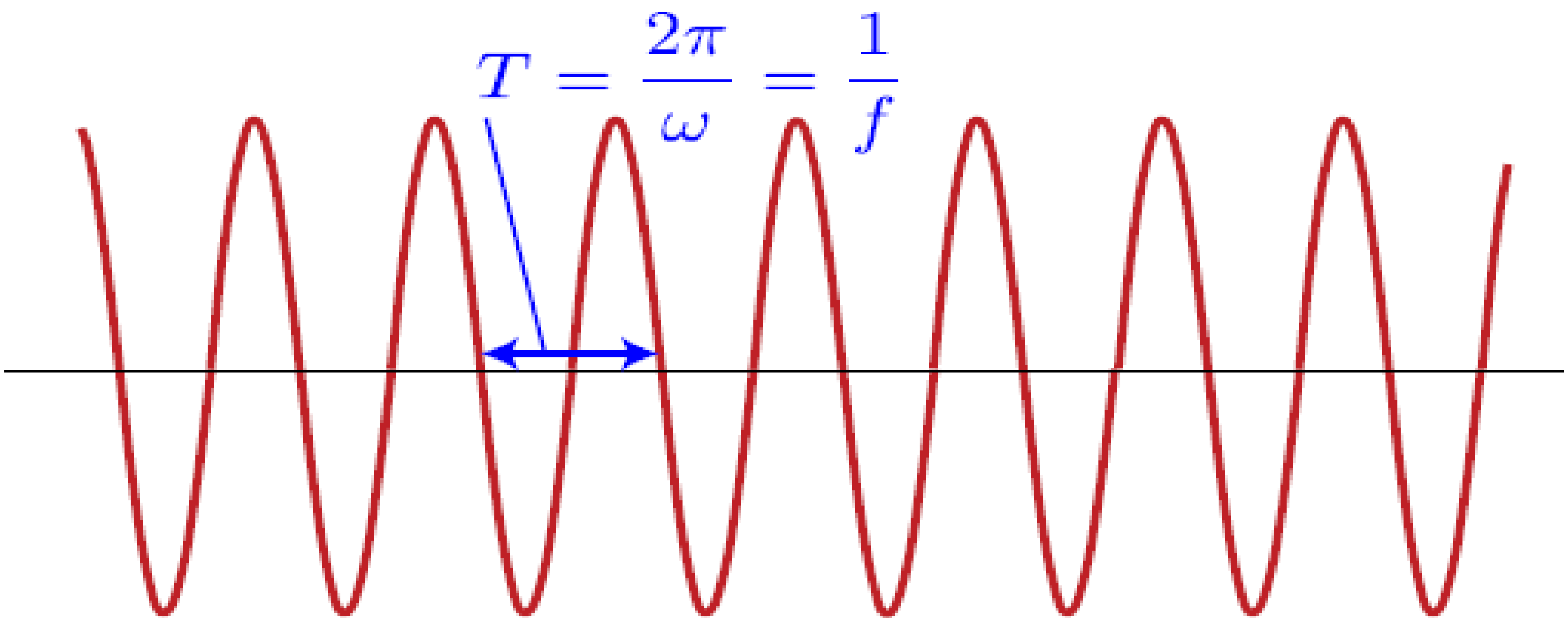
ω smaller



Classification of Signals

Discrete Time Sinusoids

ω larger, rate of oscillation higher

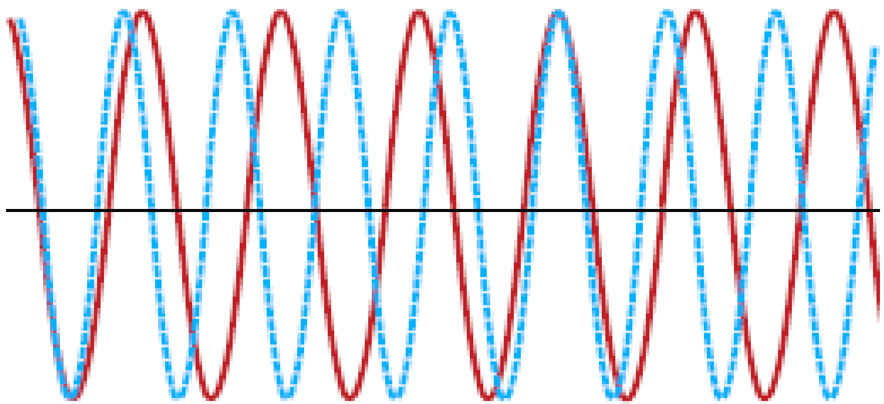


Continuous-Time Sinusoids: Frequency and Rate of Oscillation

Also, note that $x_1(t) \neq x_2(t)$ for all t for

$$x_1(t) = A\cos(\omega_1 t + \phi) \quad \text{and} \quad x_2(t) = A\cos(\omega_2 t + \phi)$$

when $\omega_1 \neq \omega_2$.



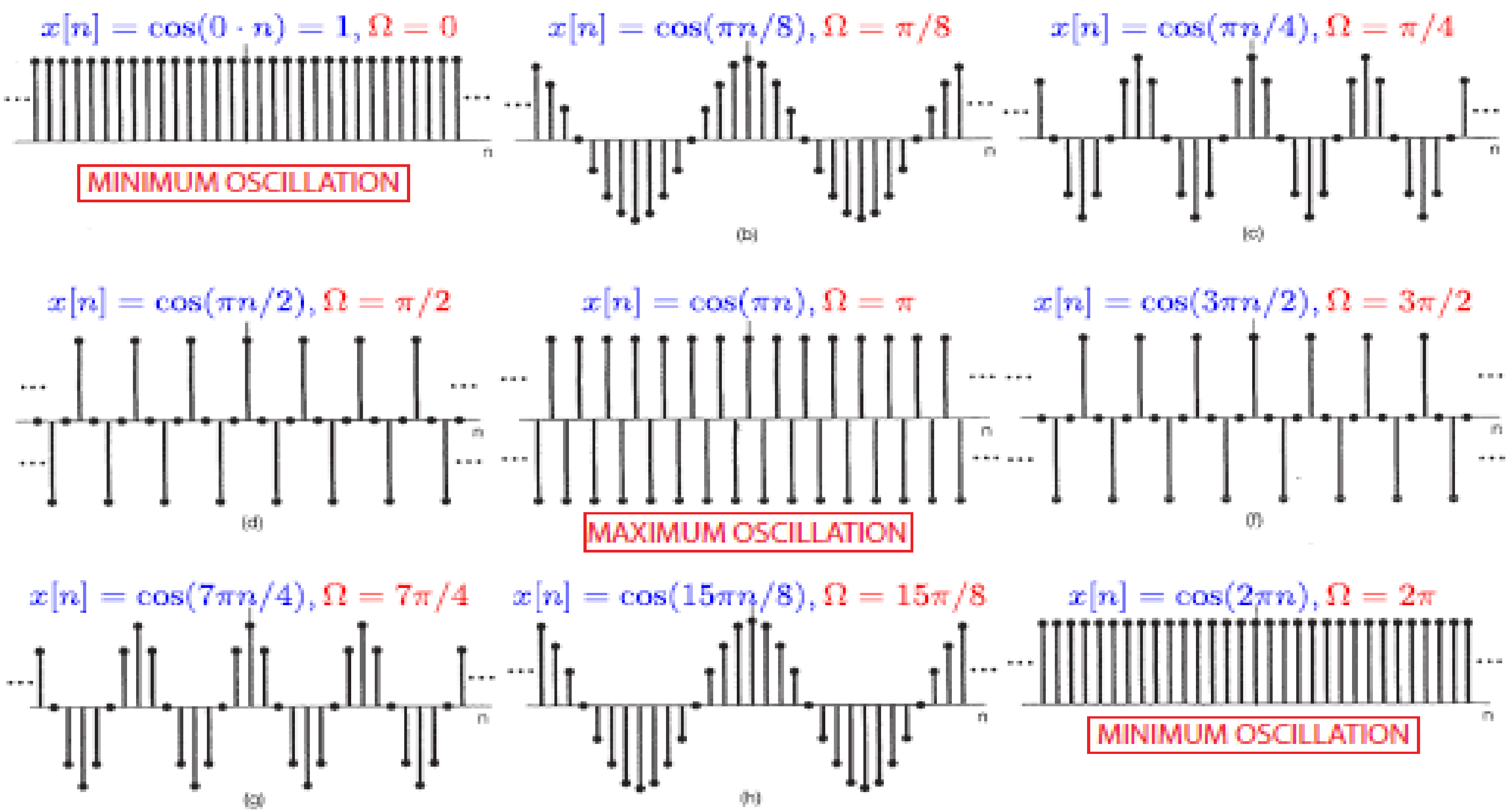
Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

Rate of oscillation increases as Ω increases UP TO A POINT then decreases again and then increases again and then decreases again . . . .

Classification of Signals

Discrete Time Sinusoids



Discrete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

Discrete-time sinusoids repeat as Ω increases!

Discrete-Time Sinusoids: Frequency and Rate of Oscillation

Let

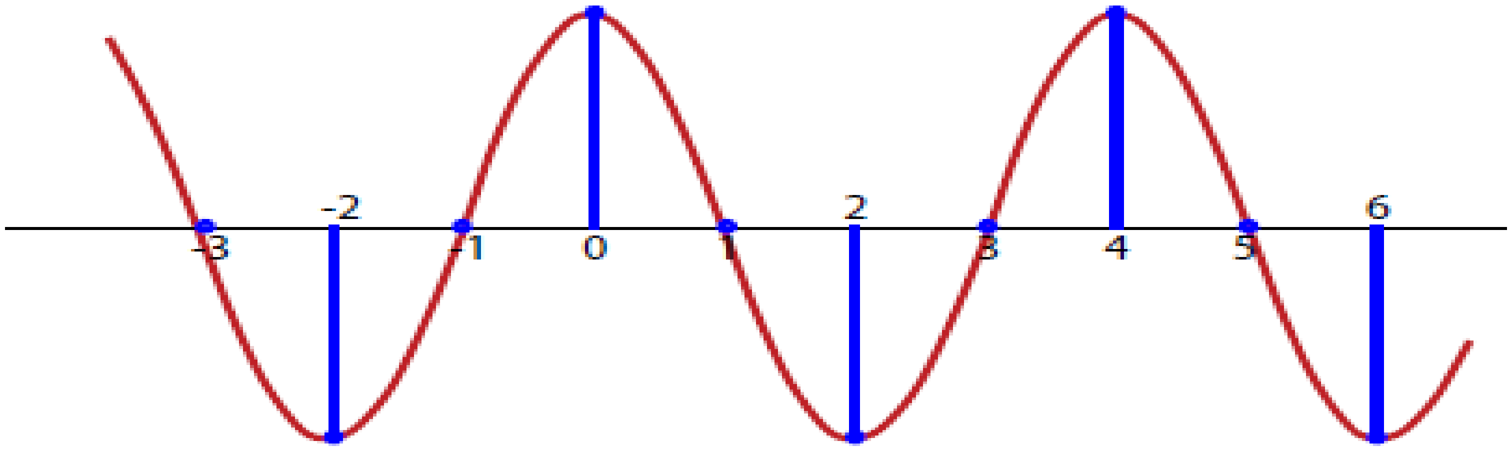
$$x_1[n] = A\cos(\Omega_1 n + \phi) \quad \text{and} \quad x_2[n] = A\cos(\Omega_2 n + \phi)$$

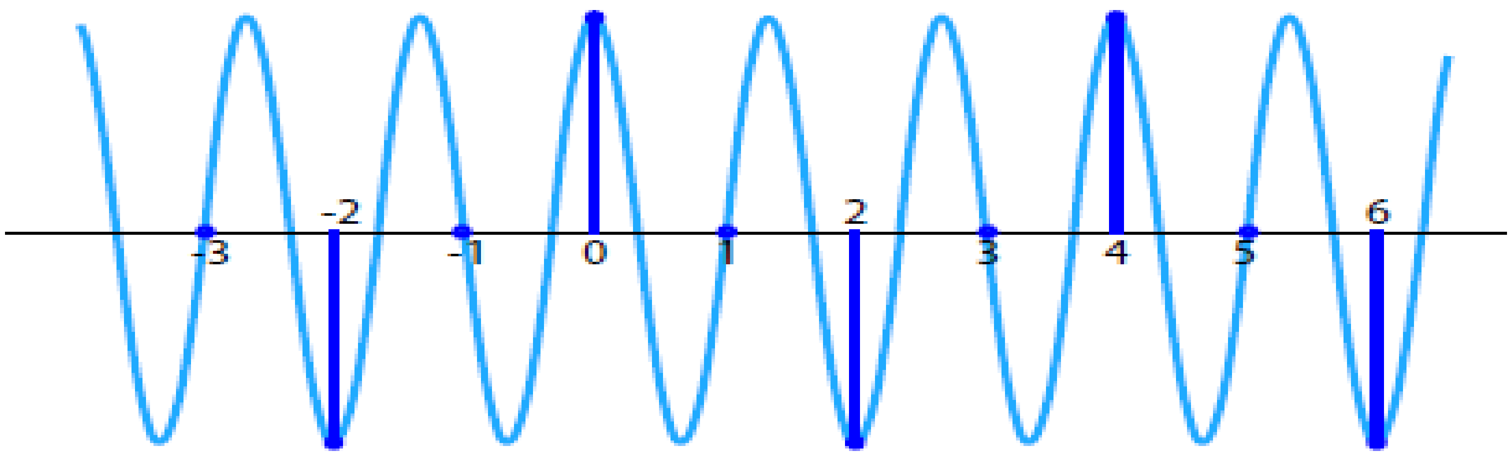
and $\Omega_2 = \Omega_1 + 2\pi k$ where $k \in \mathbb{Z}$:

$$\begin{aligned} x_2[n] &= A\cos(\Omega_2 n + \phi) \\ &= A\cos((\Omega_1 + 2\pi k)n + \phi) \\ &= A\cos(\Omega_1 n + 2\pi k n + \phi) \\ &= A\cos(\Omega_1 n + \phi) = x_1[n] \end{aligned}$$

Classification of Signals

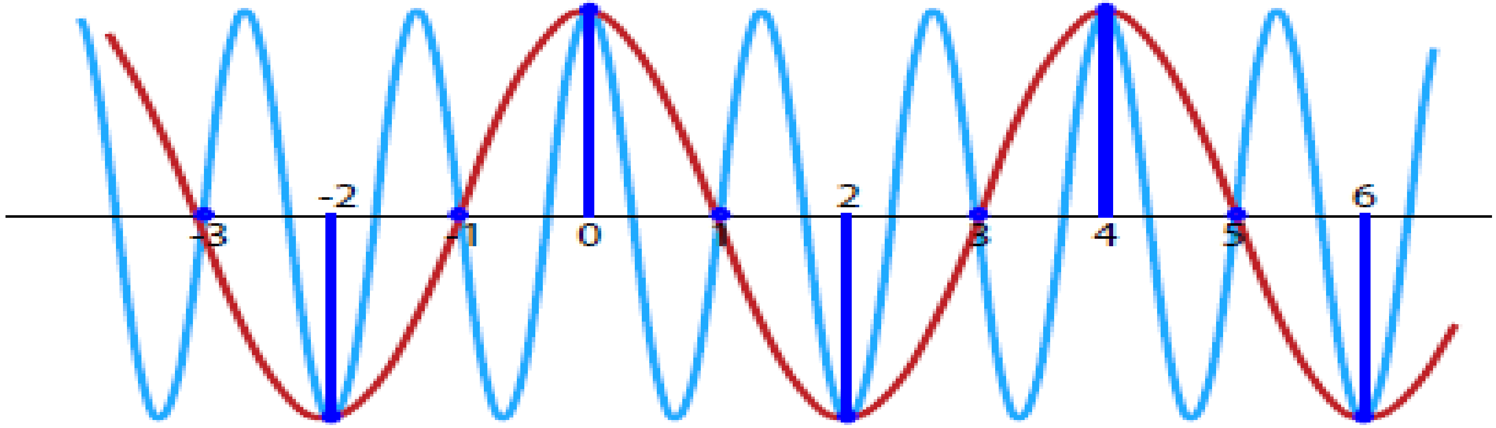
Discrete Time Sinusoids

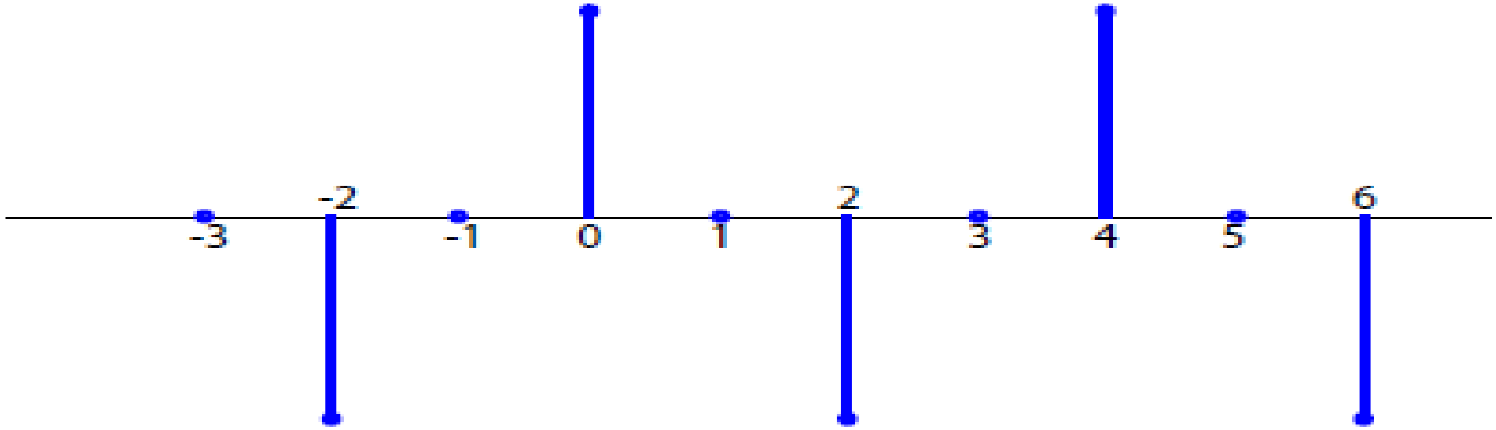




Classification of Signals

Discrete Time Sinusoids





Discete-Time Sinusoids: Frequency and Rate of Oscillation

$$x[n] = A\cos(\Omega n + \phi)$$

can be considered a sampled version of

$$x(t) = A\cos(\Omega t + \phi)$$

at integer time instants.

As Ω increases, the samples miss the faster oscillatory behavior.

Classification of Signals

- **Peroidic and Aperoidic Signals**

Example 2: Determine which of the sinusoids are periodic and compute their fundamental period.

(a) $\cos 0.01\pi n$

Solution 2:

$$\cos(0.01\pi n) = \cos\left(2\pi \times \frac{0.01}{2} n\right) = \cos\left(2\pi \frac{1}{200} n\right)$$

which means that the signal is periodic with f = 1/200 and fundamental period N = 200.

(b) $\cos(\pi 30n/105)$

Solution:

$$\cos\left(\pi \frac{30}{105} n\right) = \cos\left(2\pi \frac{30}{105\times 2} n\right) = \cos\left(2\pi \frac{1}{7} n\right)$$

i.e. the signal is periodic with f = 1/7 and fundamental period = 7.

Classification of Signals

- **Peroidic and Aperoidic Signals**

Tutorials 1:

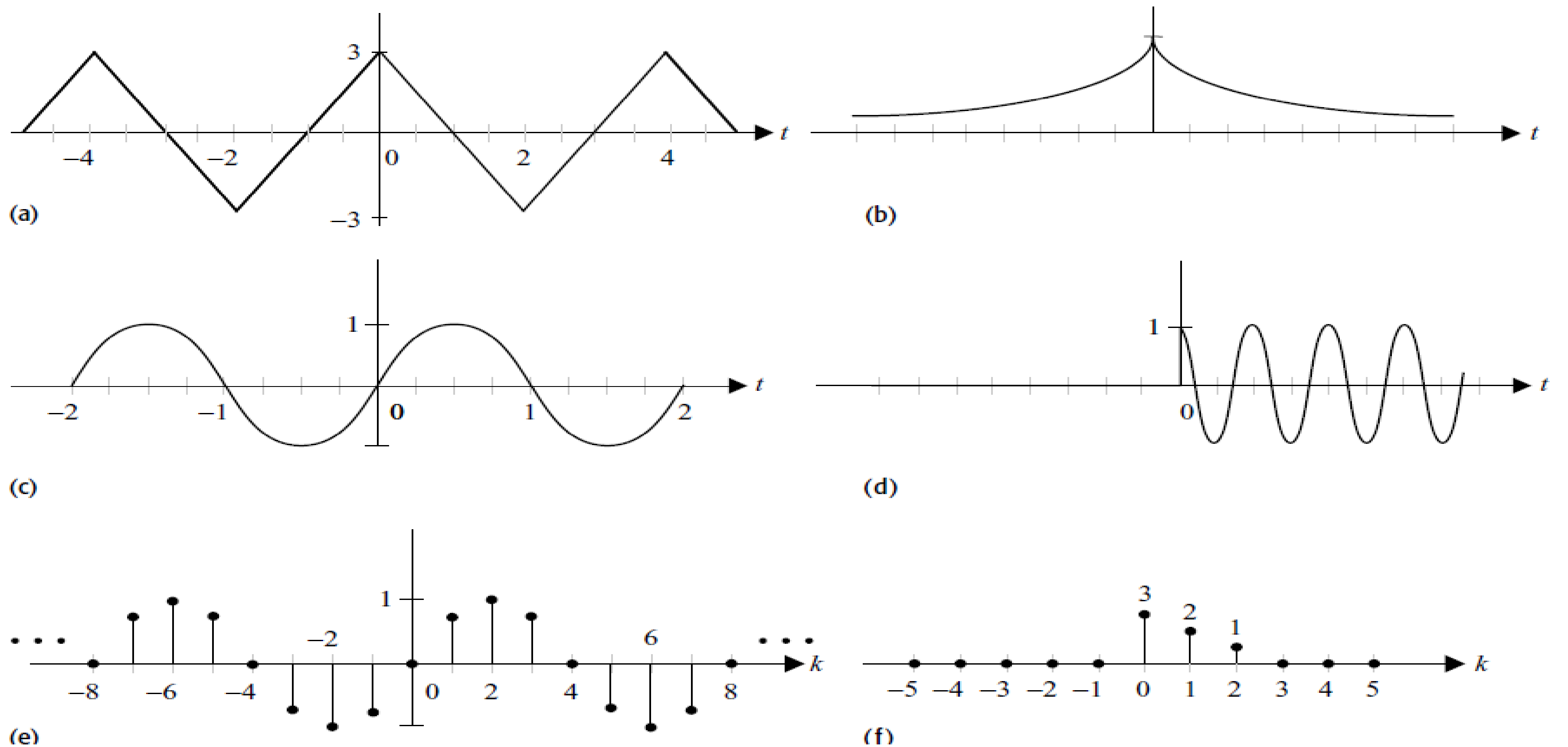
(a) $\cos(3n)$

(b) $3\cos(5n + \pi/6)$

(c) $x[n] = \cos(\pi n/2) - \sin(\pi n/8) + 3\cos(\pi n/4 + \pi/3)$

Classification of Signals

- **Peroidic and Aperoidic Signals**



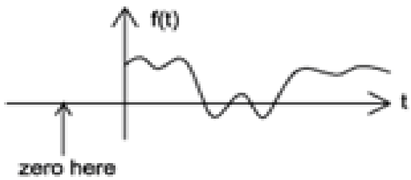
Classification of Signals

- **Causal Vs. Anticausal Vs. Noncausal**

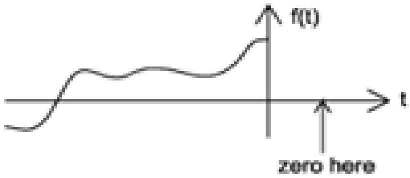
Causal Vs Anticausal Vs Noncausal

- **Causal signals** are signal that are zero for all negative time.
- **Anticausal** are signals that are zero for all positive time.
- **Noncausal** are signals that have nonzero values in both positive & negative time.

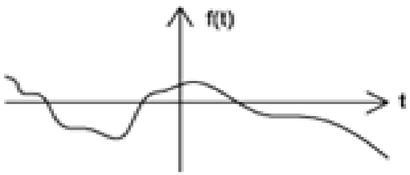
A causal signal



An anticausal signal



A noncausal signal



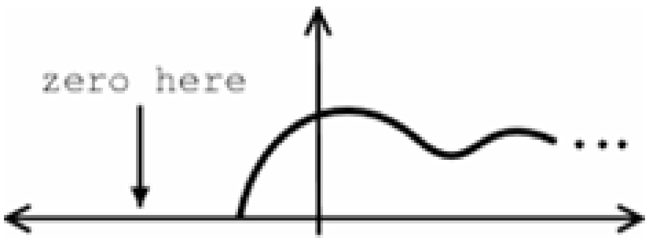
Classification of Signals

- **Right Handed Vs. Left Handed**

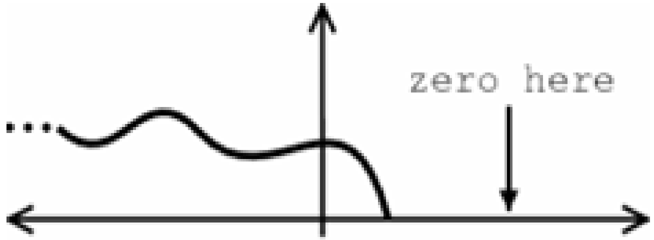
Right Handed Vs Left Handed

- **Right handed** signal is defined as any signal where $x(n) = 0$ for $n<N<\infty$.
- **Left handed** signal is defined as any signal where $x(n) = 0$ for $n>N>\infty$.

Right-handed signal



Left-handed signal

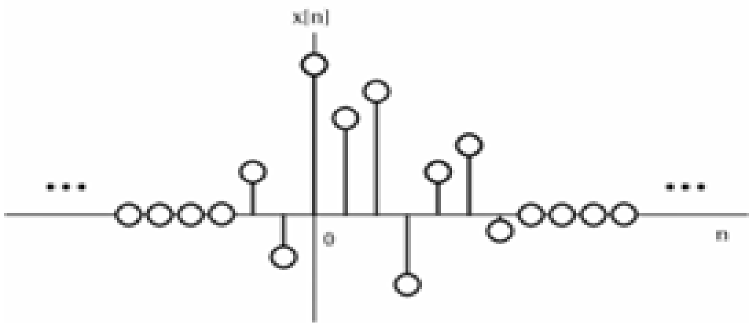


Classification of Signals

- **Finite Vs. Infinite Length**

Finite Vs Infinite Length

- Signals can be characterized as to whether they have finite or infinite length set of values.
- Most finite length signals are used when dealing with discrete time signals or a given sequence of length.
- Mathematically speaking, x(t) is a finite length signal if it is nonzero over a finite interval $t_1 < x(t) < t_2$ where $t_1 > -\infty$ & $t_2 < \infty$
- Infinite length signal, x(t), is defined as non zero over all real numbers.



- **Even Vs. Odd**

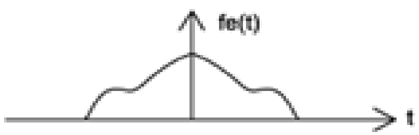
Even Vs Odd

- An **even** or symmetric signal (discrete or continuous) is any signal such that x (-t) = x(t) or x [-n] = x[n]
  - Even signals can be easily spotted as they are symmetric around the vertical axis.
- An **Odd** signal (discrete or continuous), on the other hand, is a signal such that x (-t)= -x(t) or x [-n]= -x[n]
- An odd signal is anti-symmetric!
- Any signal can be written as:

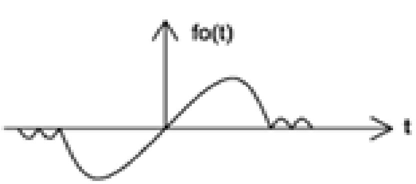
$$x(n) = x_e(n) + x_o(n)$$

Where,
$$\begin{cases} x_e(n) = \frac{1}{2}[x(n) + x(-n)] \\ x_o(n) = \frac{1}{2}[x(n) - x(-n)] \end{cases}$$

Even signal



Odd signal

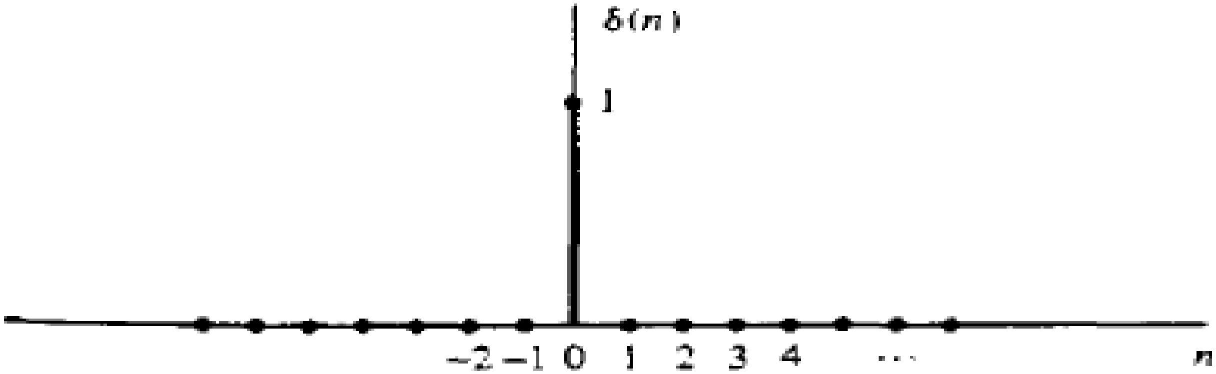


Basic Digital Sequences (Signals)

Unit impulse (unit sample)

$$\delta(n) \equiv \begin{cases} 1, & n = 0, \\ 0, & n \neq 0 \end{cases} .$$

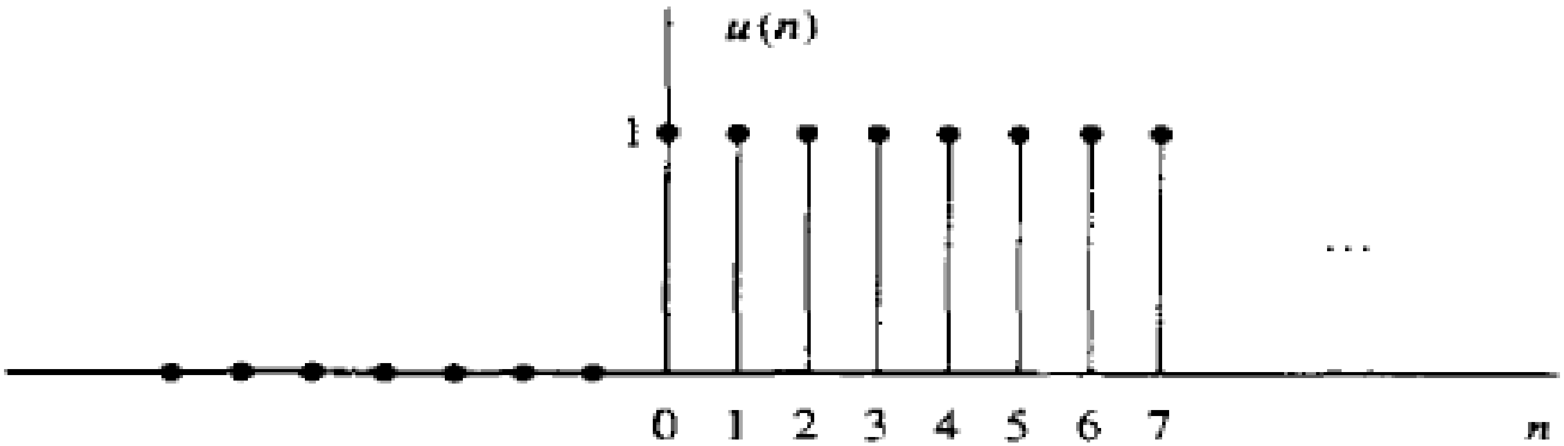
$$\begin{aligned} u(n) &= \sum_{m=-\infty}^{n} \delta(m) \quad \text{summing}, \\ \delta(n) &= u(n) - u(n-1) \quad \text{differencing}. \end{aligned}$$



Basic Digital Sequences (Signals)

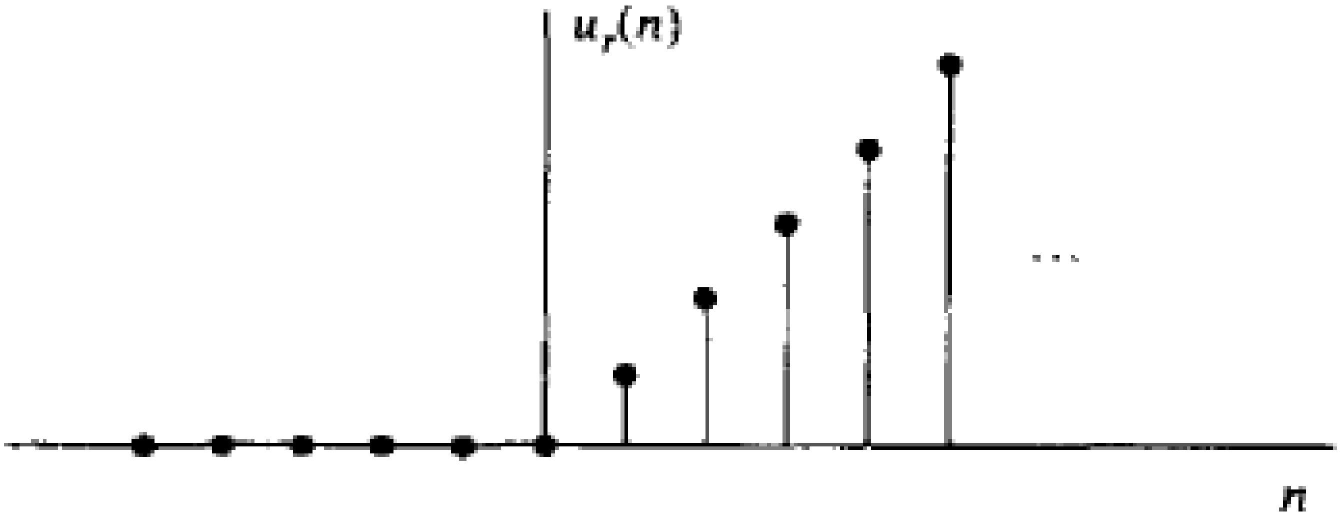
Unit step signal

$$u(n) \equiv \begin{cases} 1, & n \geq 0, \\ 0, & n < 0 \end{cases}$$



Unit Ramp Signal

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

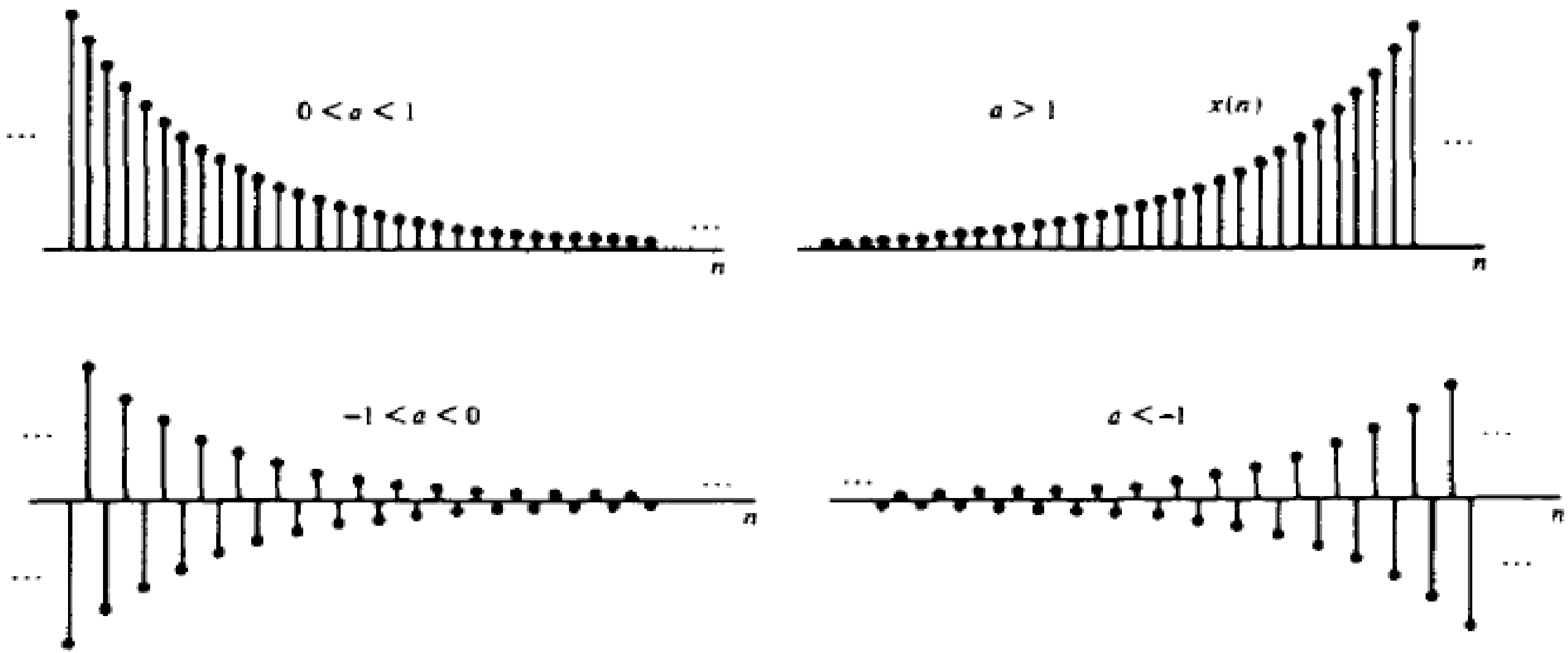


Basic Digital Sequences (Signals)

Exponential Signal

$$x(n) = a^n \qquad \text{for all } n$$

If the parameter a is real, then $x(n)$ is a real signal.



Exponential Signal

$$x(n) = a^n \qquad \text{for all } n$$

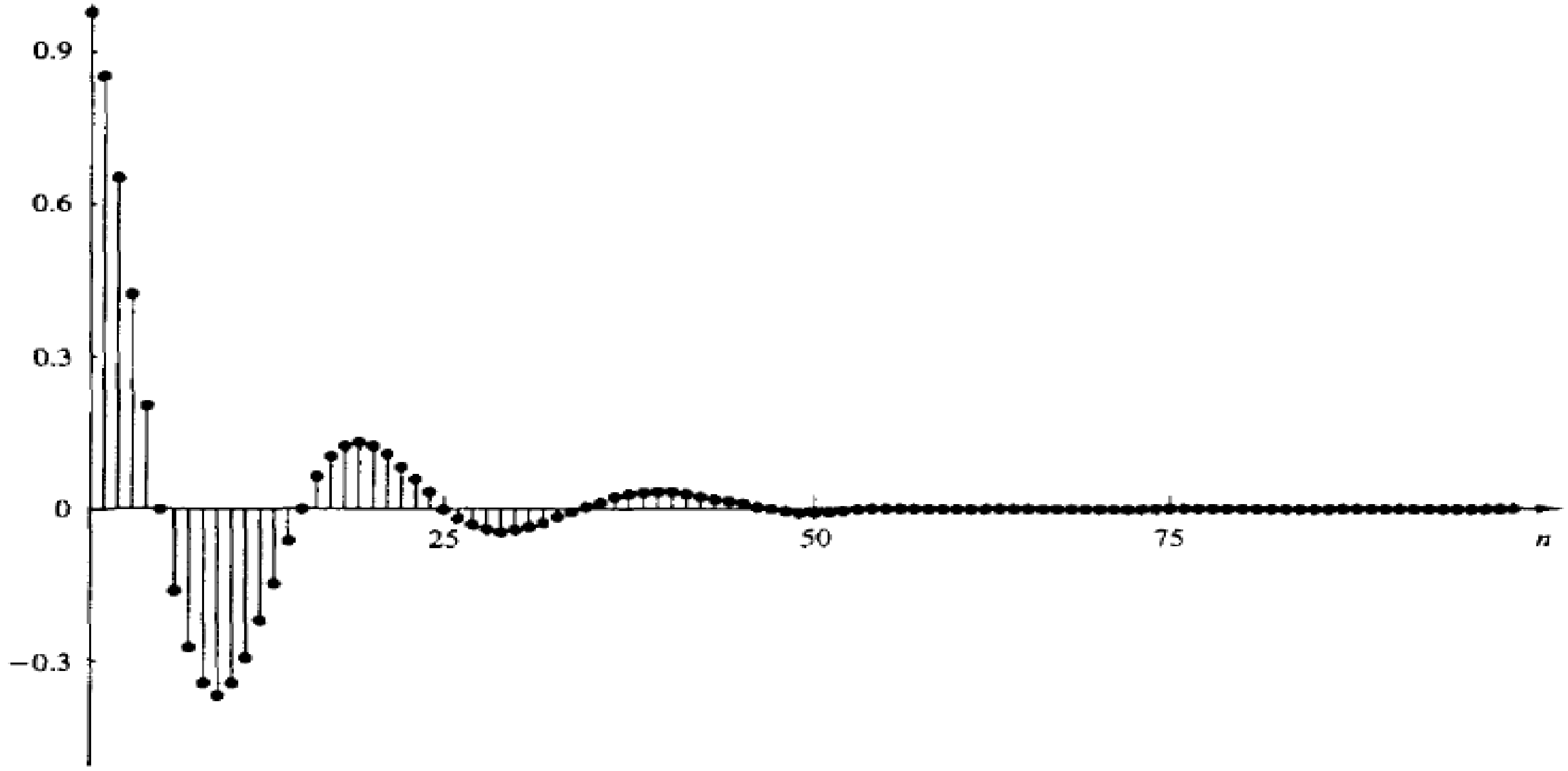
When the parameter a is complex valued, it can be expressed as

$$a \equiv re^{j\theta}$$

where r and θ are now the parameters. Hence we can express $x(n)$ as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n(\cos\theta n + j\sin\theta n) \end{aligned}$$

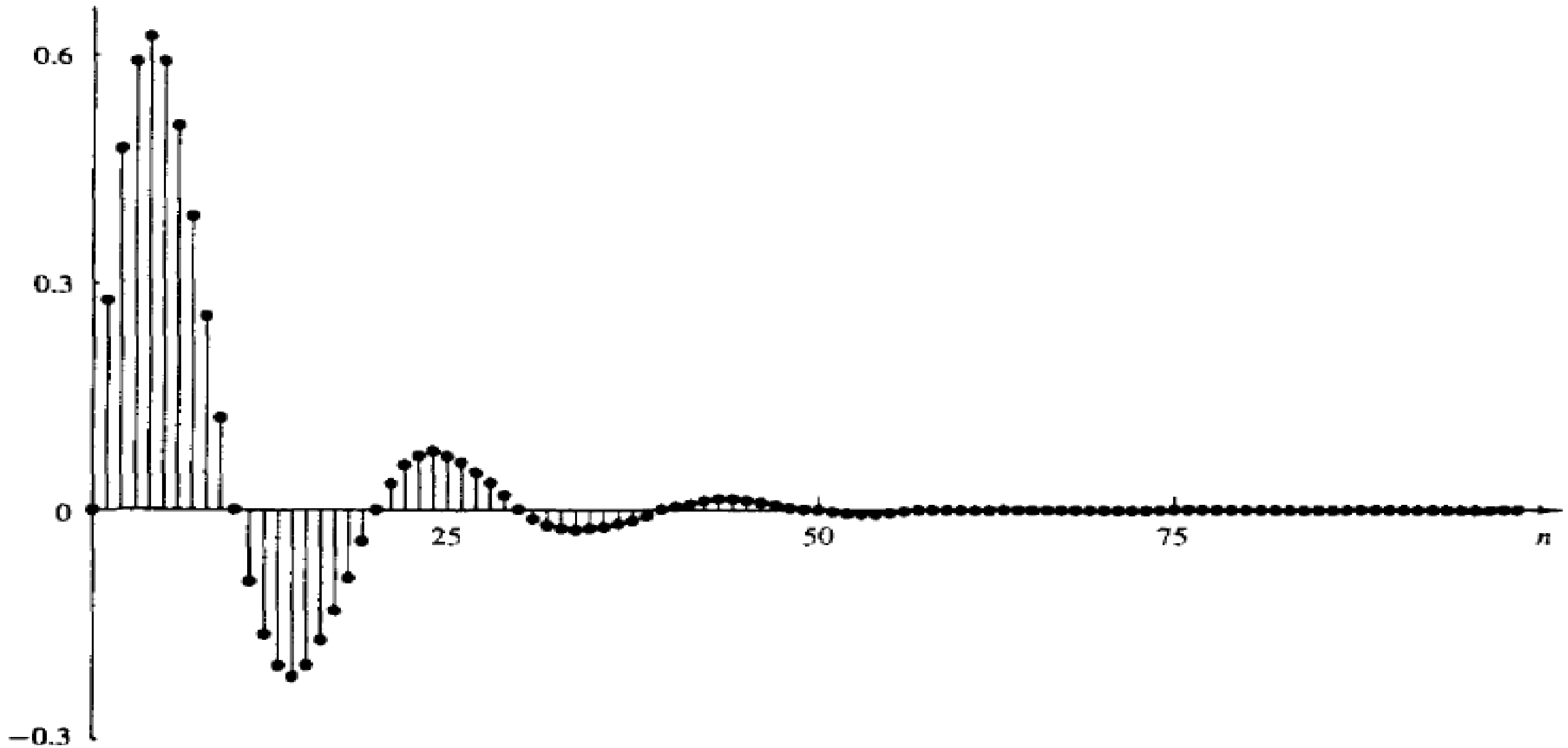
Exponential Signal



(a) Graph of $x_R(n) = (0.9)^n \cos \frac{\pi n}{10}$

Basic Digital Sequences (Signals)

Exponential Signal



(b) Graph of $x_f(n) = (0.9)^n \sin \frac{\pi n}{10}$

Sinusoids Signal

Sinusoids

$$x(n) = A\sin(\omega n + \theta)$$

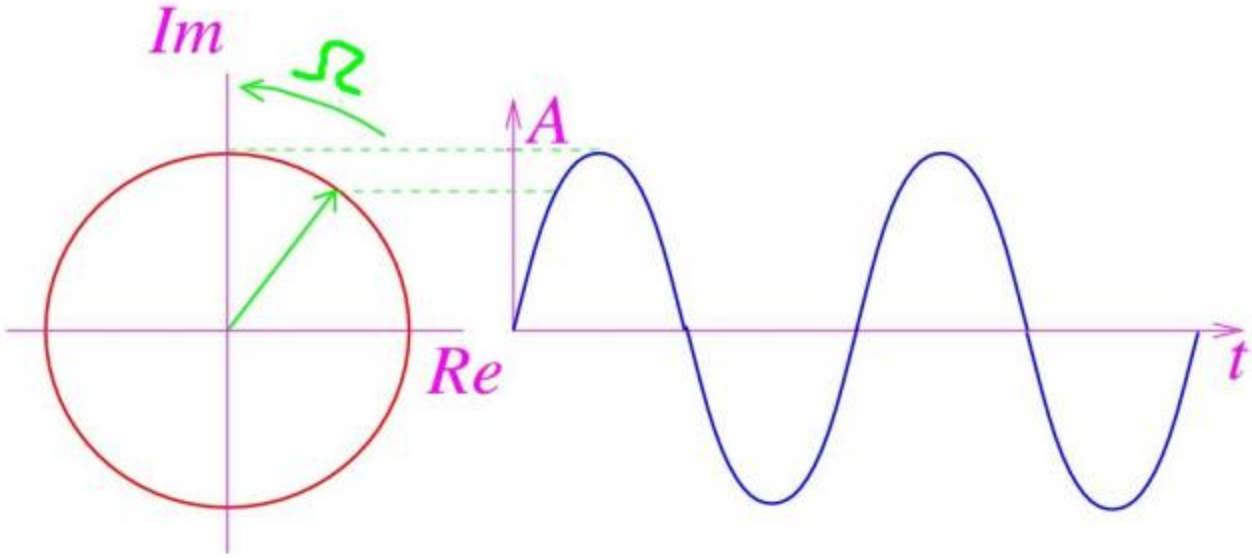
Useful properties:

$$\begin{aligned}
\exp[j(\omega n + \theta)] &= \cos(\omega n + \theta) + j\sin(\omega n + \theta), \\
\cos(\omega n + \theta) &= \frac{\exp[j(\omega n + \theta)] + \exp[-j(\omega n + \theta)]}{2}, \\
\sin(\omega n + \theta) &= \frac{\exp[j(\omega n + \theta)] - \exp[-j(\omega n + \theta)]}{2j}.
\end{aligned}$$

Basic Digital Sequences (Signals)

Sinusoids Signal

A sine wave as the projection of a complex phasor onto the imaginary axis:



Classification of DSP Systems

Linear Vs. Non-linear Systems

A **linear** system is any system that obeys the properties of scaling (homogeneity) and superposition (additivity), while a **nonlinear** system is any system that does not obey at least one of these.

To show that a system H obeys the scaling property is to show that

$$H\left(kf\left(t\right)\right) = kH\left(f\left(t\right)\right)$$

To demonstrate that a system H obeys the superposition property of linearity is to show that

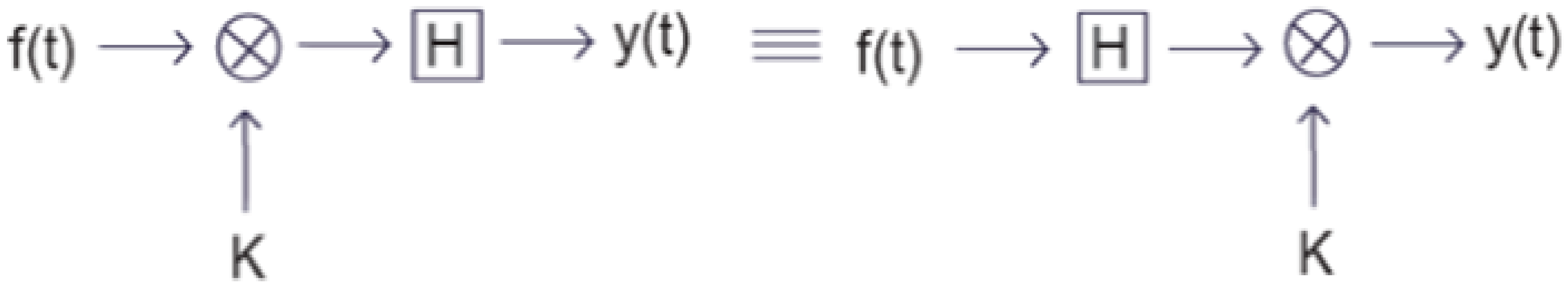
$$H\left(f_1\left(t\right) + f_2\left(t\right)\right) = H\left(f_1\left(t\right)\right) + H\left(f_2\left(t\right)\right)$$

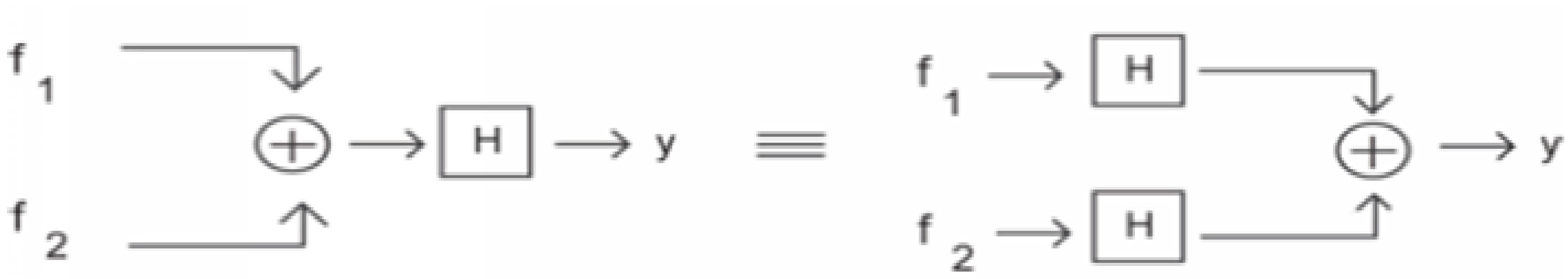
It is possible to check a system for linearity in a single (though larger) step. To do this, simply combine the first two steps to get

$$H\left(k_1 f_1\left(t\right) + k_2 f_2\left(t\right)\right) = k_2 H\left(f_1\left(t\right)\right) + k_2 H\left(f_2\left(t\right)\right)$$

Classification of DSP Systems

Linear vs. Non-linear Systems





Linear vs. Non-linear Systems

- Linear System: A system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} \quad \text{(additivity)}$$

and

$$T\{ax[n]\} = aT\{x[n]\} \quad \text{(scaling)}$$

- Examples
  - Ideal Delay System

$$y[n] = x[n - n_o]$$

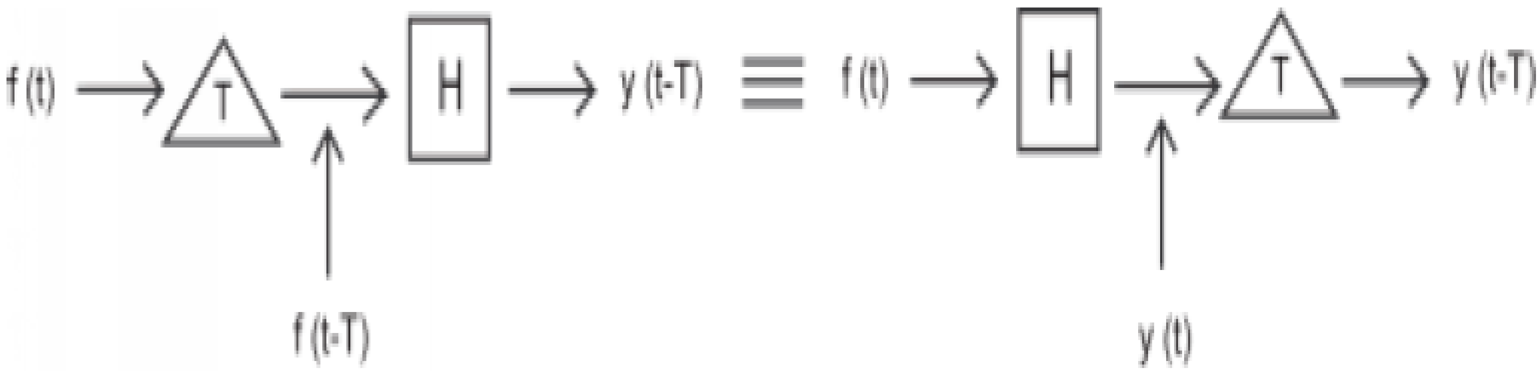
$$\begin{aligned} T\{x_1[n] + x_2[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\ T\{x_2[n]\} + T\{x_1[n]\} &= x_1[n - n_o] + x_2[n - n_o] \\ T\{ax[n]\} &= ax_1[n - n_o] \\ aT\{x[n]\} &= ax_1[n - n_o] \end{aligned}$$

Classification of DSP Systems

Time Invariant vs. Time Variant

A **time invariant** system is one that does not depend on when it occurs: the shape of the output does not change with a delay of the input. That is to say that for a system H where $H(f(t)) = y(t)$, H is time invariant if for all T

$$H(f(t-T)) = y(t-T)$$



When this property does not hold for a system, then it is said to be **time variant** , or time-varying.

Classification of DSP Systems

Time Invariant vs. Time Variant

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output

$$y[n] = T\{x[n]\} \Rightarrow y[n-n_o] = T\{x[n-n_o]\}$$

- Example
  - Square

$y[n] = (x[n])^2$

Delay the input the output is $y_1[n] = (x[n-n_o])^2$

Delay the output gives $y[n - n_o] = (x[n-n_o])^2$

- Counter Example
  - Compressor System

$y[n] = x[Mn]$

Delay the input the output is $y_1[n] = x[Mn-n_o]$

Delay the output gives $y[n - n_o] = x[M(n-n_o)]$

Classification of DSP Systems

Casual vs. Noncasual

A **causal** system is one that is **nonanticipative** ; that is, the output may depend on current and past inputs, but not future inputs. All "realtime" systems must be causal, since they can not have future inputs available to them.

One may think the idea of future inputs does not seem to make much physical sense; however, we have only been dealing with time as our dependent variable so far, which is not always the case. Imagine rather that we wanted to do image processing. Then the dependent variable might represent pixels to the left and right (the "future") of the current position on the image, and we would have a **noncausal** system.

Causality
- A system is causal it's output is a function of only the current and previous samples

Examples
- Backward Difference

$$y[n] = x[n] - x[n-1]$$

Counter Example
- Forward Difference

$$y[n] = x[n+1] + x[n]$$

Classification of DSP Systems

Stable vs. Nonstable

A **stable** system is one where the output does not diverge as long as the input does not diverge. A bounded input produces a bounded output. It is from this property that this type of system is referred to as **bounded input-bounded output (BIBO)** stable.

Representing this in a mathematical way, a stable system must have the following property, where $x(t)$ is the input and $y(t)$ is the output. The output must satisfy the condition

$$|y(t)| \leq M_y < \infty$$

when we have an input to the system that can be described as

$$|x(t)| \leq M_x < \infty$$

M_x and M_y both represent a set of finite positive numbers and these relationships hold for all of t.

If these conditions are not met, i.e. a system's output grows without limit (diverges) from a bounded input, then the system is **unstable** .

Classification of DSP Systems

Stable vs. Nonstable

Stability (in the sense of bounded-input bounded-output BIBO)

- A system is stable if and only if every bounded input produces a bounded output

$$|x[n]| \leq B_x < \infty \Rightarrow |y[n]| \leq B_y < \infty$$

Example

- Square

$$y[n] = (x[n])^2$$

if input is bounded by $|x[n]| \leq B_x < \infty$

output is bounded by $|y[n]| \leq B_x^2 < \infty$

Counter Example

- Log

$$y[n] = \log_{10}(|x[n]|)$$

even if input is bounded by $|x[n]| \leq B_x < \infty$

output not bounded for $x[n] = 0 \Rightarrow y[0] = \log_{10}(|x[n]|) = -\infty$

Memoryless System

- Memoryless System
  - A system is memoryless if the output y[n] at every value of n depends only on the input x[n] at the same value of n

- Example Memoryless Systems
  - Square

$$y[n] = (x[n])^2$$

  - Sign

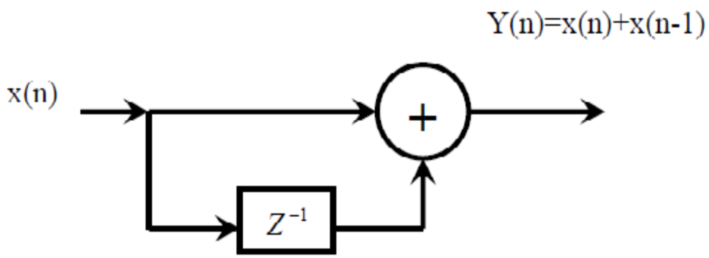
$$y[n] = \text{sign}\{x[n]\}$$

- Counter Example
  - Ideal Delay System

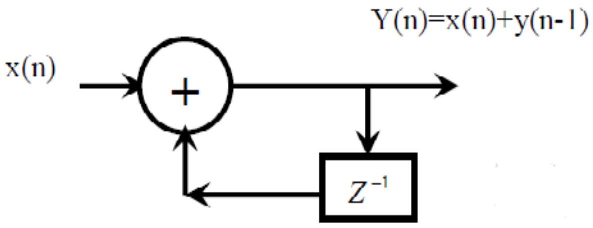
$$y[n] = x[n - n_o]$$

Recursive and Nonrecursive Digital Filters

A recursive system is one in which the output y(n) is dependent on one or more of its past outputs (y(n-1), y(n-2)G) while a non recursive system is one in which the output is independent of any past outputs .e.g. feedforward system having no feedback is a non recursive system.



Non Recursive System



Recursive System