<u>**Discrete Memoryless Channel (DMC):**</u>

15.8.1. Channel Representation

A communication channel may be defined as the path or medium through which the symbols flow to the receiver end. A *discrete memoryless channel* (DMC) is a statistical model with an input X and an output Y as shown in figure 15.1. During each unit of the time (signaling interval), the channel accepts an input symbol from X, and in response it generates an output symbol from Y. The channel is said to be "discrete" when the alphabets of X and Y are both finite. Also, it is said to be "memoryless" when the current output depends on only the current input and not on any of the previous inputs.

A diagram of a DMC with m inputs and n outputs has been illustrated in figure 15.1. The input X consists of input symbols $x_1, x_2, \ldots x_m$. The a priori probabilities of these source symbols $P(x_i)$ are assumed to be known. The outputs Y consists of output symbols $y_1, y_2, \ldots y_n$. Each possible input-to-output path is indicated along with a conditional probability $P(y_j \mid x_i)$, where $P(y_j \mid x_i)$ is the conditional probability of obtaining output y_j given that the input is x_i, and is called a **channel transition probability.**

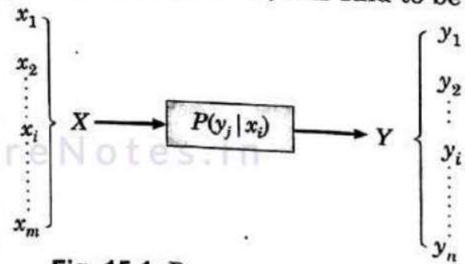

Fig. 15.1 *Representation of a discrete memoryless channel (DMC).*

15.8.2. The Channel Matrix

A channel is completely specified by the complete set of transition probabilities. Accordingly, the channel in figure 15.1 is often specified by the matrix of transition probabilities $[P(Y \mid X)]$. This matrix is given by

$$[P(Y \mid X)] = \begin{bmatrix} P(y_1 \mid x_1) & P(y_2 \mid x_1) & \cdots & P(y_n \mid x_1) \\ P(y_1 \mid x_2) & P(y_2 \mid x_2) & \cdots & P(y_n \mid x_2) \\ \cdots & \cdots & \cdots & \cdots \\ P(y_1 \mid x_m) & P(y_2 \mid x_m) & \cdots & P(y_n \mid x_m) \end{bmatrix} \quad \ldots(15.13)$$

This matrix $[P(Y \mid X)]$ is called the **channel matrix.**

Since each input to the channel results in some output, each row of the channel matrix must sum to unity. This means that

$$\sum_{j=1}^{n} P(y_j \mid x_i) = 1 \text{ for all } i \quad \ldots(15.14)$$

Now, if the input probabilities $P(X)$ are represented by the row matrix, then we have

$$[P(X)] = [P(x_1) \;\; P(x_2) \ldots P(x_m)] \quad \ldots(15.15)$$

Also, the output probabilities $P(Y)$ are represented by the row matrix as under:

$$[P(Y)] = [P(y_1) \;\; P(y_2) \ldots P(y_n)] \quad \ldots(15.16)$$

then

$$[P(Y)] = [P(X)][P(Y \mid X)] \quad \ldots(15.17)$$

Now, if $P(X)$ is represented as a diagonal matrix, then we have

$$[P(X)]_d = \begin{bmatrix} P(x_1) & 0 & \cdots & 0 \\ 0 & P(x_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & P(x_m) \end{bmatrix} \quad ...(15.18)$$

then $$[P(X,Y)] = [P(X)]_d \, [P(Y|X)] \quad ...(15.19)$$

where the (i, j) element of matrix $[P(X,Y)]$ has the form $P(x_i, y_j)$.

The matrix $[P(X, Y)]$ is known as the *joint probability matrix*, and the element $P(x_i, y_j)$ is the joint probability of transmitting x_i and receiving y_j.

15.9.1. Lossless Channel

A channel described by a channel matrix with only one non-zero element in each column is called a *lossless channel*. An example of a lossless channel has been shown in figure 15.2, and the corresponding channel matrix is given in equation (15.20) as under:

$$[P(Y|X)] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ...(15.20)$$

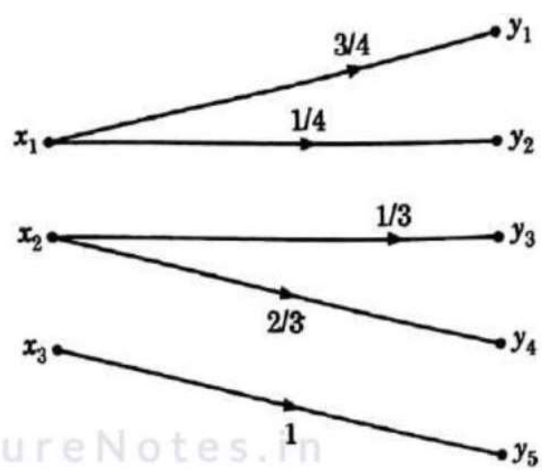

Fig1: Noiseless Channel

It can be shown that in the lossless channel, no source information is lost in transmission.

15.9.2. Deterministic Channel

A channel described by a channel matrix with only one non-zero element in each row is called a *deterministic channel*. An example of a deterministic channel has been shown in figure 15.3, and the corresponding channel matrix is given by equation (15.21) as under:

$$[P(Y|X)] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{...(15.21)}$$

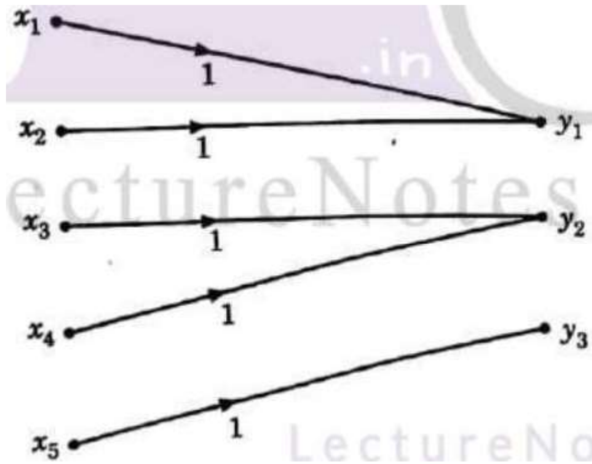

Fig2: Deterministic Channel

Important Point: *It may be noted that since each row has only one non-zero element, therefore, this element must be unity by equation (15.14). Thus , when a given source symbol is sent in the deterministic channel, it is clear which output symbol will be received.*

15.9.3. Noiseless Channel

A channel is called *noiseless* if it is both lossless and deterministic. A noiseless channel has been shown in figure 15.4. The channel matrix has only one element in each row and in each column, and this element is unity. Note that the input and output alphabets are of the same size, that is, $m = n$ for the noiseless channel.

The matrix for a noiseless channel is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

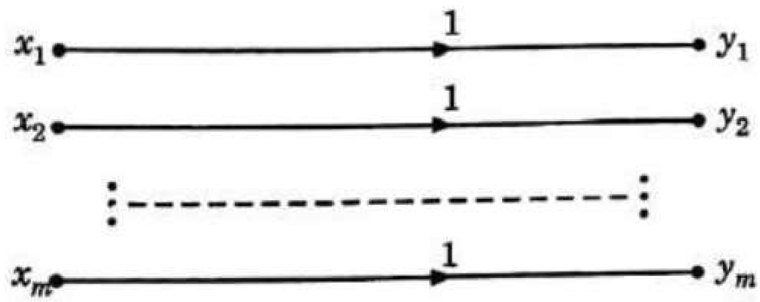

Fig2: Noiseless Channel

15.9.4. Binary Symmetric Channel (BSC)

The binary symmetric channel (BSC) is defined by the channel diagram shown in figure 15.5, and its channel matrix is given by

$$[P(Y|X)] = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \quad \text{...(15.22)}$$

A BSC channel has two inputs ($x_1 = 0, x_2 = 1$) and two outputs ($y_1 = 0, y_2 = 1$). This channel is symmetric because the probability of receiving a 1 if a 0 is sent is the same as the probability of receiving a 0 if a 1 is sent. This common transition probability is denoted by p as shown in figure 15.5.

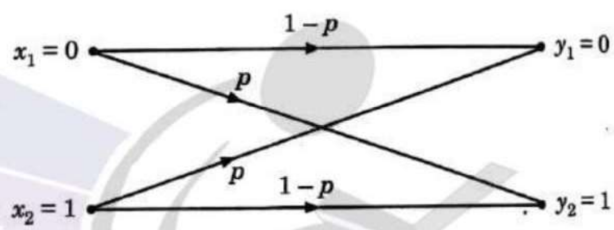

Fig. 15.5 *Binary symmetrical channel.*

The binary symbols '0' and '1' are transmitted with probabilities 1/4 and 3/4 respectively. Find the corresponding Self Informations.

: Self Information in a '0' $= I_0 = \log \frac{1}{P_0} = \log 4$

$$\Rightarrow I_0 = \frac{\log 4}{\log 2} = 2 \text{ bits}$$

$$\therefore \boxed{I_0 = 2 \text{ bits}}$$

Self Information in a '1' $= I_1 = \log \frac{1}{P_1} = \log_2\left(\frac{4}{3}\right)$

$$\Rightarrow I_1 = \frac{\log_{10} \frac{4}{3}}{\log_{10} 2}$$

$$\therefore \boxed{I_1 = 0.415 \text{ bits}}$$

Thus it can be observed that more information is carried by a less likely message.

3> If there are M equally likely and independent messages, then Prove that amount of information carried by each message will be $I = N$ bits, where $M = 2^N$ and N is an integer.

Sol: Since all the messages are equally likely and independent, probability of occurrence of each message will be $\frac{1}{M}$.

We know that, the amount of information is given by,

$$I_K = \log_2 \frac{1}{P_K}$$

Since $P_K = \frac{1}{M}$ then $I_K = \log_2 M$

As wkt, $M = 2^N$, Hence above equation becomes,

$$I_K = \log_2 2^N$$

$$= N \log_2 2 = N \frac{\log_{10} 2}{\log_{10} 2}$$

$$\Rightarrow \boxed{I_K = N \text{ bits}}$$ Hence the proof.

Prove the following statement, "If receiver knows the message being transmitted, the amount of information carried is "Zero".

: Here it is stated that receiver "knows" the message. This means only one message is transmitted. Hence probability of occurrence of this message will be $P_K = 1$.

Therefore, the amount of information carried by this type of message is,

$$I_K = \log_2\left(\frac{1}{P_K}\right)$$

$$= \frac{\log_{10} 1}{\log_{10} 2}$$

$$\therefore \boxed{I_K = 0 \text{ bits}}$$

INFORMATION RATE :

Let us suppose that, the symbols are emitted by the source at a fixed time rate "r_s" symbols/sec. The average source information rate R_s" in bits/sec is defined as the product of the average information content per symbol and the message symbol rate "r_s".

$$\therefore \quad \boxed{R_s = r_s \, H(S)} \quad \text{bits/sec or BPS}$$

Consider a source $S = \{S_1, S_2, S_3\}$ with $P = \{1/2, 1/4, 1/4\}$
Find (a) Self Information of each message

(b) Entropy of source 'S'.

a) Self Information of $S_1 = I_1 = \log_2 \frac{1}{P_1} = \log_2 2 = 1$ bits

Self Information of $S_2 = I_2 = \log_2 \frac{1}{P_2} = \log_2 4 = 2$ bits

Self Information of $S_3 = I_3 = \log_2 \frac{1}{P_3} = \log_2 4 = 2$ bits

b). Average Information content or Entropy is given by,

$$H(S) = \sum_{i=1}^{3} P_i I_i = P_1 I_1 + P_2 I_2 + P_3 I_3$$

$$= \left(\tfrac{1}{2}\right)(1) + \tfrac{1}{4}(2) + \tfrac{1}{4}(2)$$

$$\therefore \quad \boxed{H(S) = 1.5 \text{ bits/msg symbol}}$$

A source emits one of four symbols s_0, s_1, s_2 and s_3 with probabilities $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}$ and $\frac{1}{4}$ respectively. The successive symbols emitted by the source are statistically independent. Calculate the entropy of the source.

Sol: The entropy of the source is given by,

$$H(S) = \sum_{i=0}^{3} P_i \log \frac{1}{P_i}$$

$$= P_0 \log_2 \frac{1}{P_0} + P_1 \log_2 \frac{1}{P_1} + P_2 \log_2 \frac{1}{P_2} + P_3 \log_2 \frac{1}{P_3}$$

$$= \frac{1}{3} \log_2 3 + \frac{1}{6} \log_2 6 + \frac{1}{4} \log_2 4 + \frac{1}{4} \log_2 4$$

$$= 0.5282 + 0.43082 + 0.5 + 0.5$$

$$\therefore \boxed{H(S) = 1.95914 \text{ bits/msg symbol}}$$

A source emits one of 4 possible symbols "x_0 to x_3" during each signalling interval. The symbols occur with probabilities as given in table below:

Symbol	Probability
x_0	$P_0 = 0.4$
x_1	$P_1 = 0.3$
x_2	$P_2 = 0.2$
x_3	$P_3 = 0.1$

Find the amount of information gained by observing the source emitting each of these symbols and also the entropy of source.

Sol: The self-Information "I_k" is given by

$$I_k = \log_2 \frac{1}{P_k} \text{ bits} \qquad (1)$$

Since we have four symbols, so $K = 0, 1, 2, 3$

$\therefore$ when $K=0$, $I_0 = \log_2 \frac{1}{P_0} = \log_2 \frac{1}{0.4} = 1.322$ bits

$K=1$, $I_1 = \log_2 1/P_1 = \log_2 1/0.3 = 1.737$ bits

$K=2$, $I_2 = \log_2 1/P_2 = \log_2 1/0.2 = 2.322$ bits

$K=3$, $I_3 = \log_2 1/P_3 = \log_2 1/0.1 = 3.322$ bits

The entropy of the source is given by,

$$H(X) = \sum_{K=0}^{3} P_K \log \frac{1}{P_K} \text{ bits / msg symbol}$$

$$= \sum_{K=0}^{3} P_K I_K \qquad \because I_K = \log \frac{1}{P_K}$$

$$= P_0 I_0 + P_1 I_1 + P_2 I_2 + P_3 I_3$$

$$= 0.4(1.322) + 0.3(1.737) + 0.2(2.322) + 0.1(3.322)$$

$$\therefore \boxed{H(X) = 1.8465 \text{ bits/msg symbol}}$$