8

State-space methods for control system design

8.1 The state-space-approach

The classical control system design techniques discussed in Chapters 5–7 are generally only applicable to

(a) Single Input, Single Output (SISO) systems
(b) Systems that are linear (or can be linearized) and are time invariant (have parameters that do not vary with time).

The state-space approach is a generalized time-domain method for modelling, analysing and designing a wide range of control systems and is particularly well suited to digital computational techniques. The approach can deal with

(a) Multiple Input, Multiple Output (MIMO) systems, or multivariable systems
(b) Non-linear and time-variant systems
(c) Alternative controller design approaches.

8.1.1 The concept of state

The state of a system may be defined as: 'The set of variables (called the state variables) which at some initial time t_0, together with the input variables completely determine the behaviour of the system for time $t \geq t_0$'.

The state variables are the smallest number of states that are required to describe the dynamic nature of the system, and it is not a necessary constraint that they are measurable. The manner in which the state variables change as a function of time may be thought of as a trajectory in n dimensional space, called the *state-space*. Two-dimensional state-space is sometimes referred to as the *phase-plane* when one state is the derivative of the other.

8.1.2 The state vector differential equation

The state of a system is described by a set of first-order differential equations in terms of the state variables $(x_1, x_2, \ldots, x_n)$ and input variables $(u_1, u_2, \ldots, u_n)$ in the general form

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m
\end{aligned}
\tag{8.1}
$$

The equations set (8.1) may be combined in matrix format. This results in the state vector differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.2}$$

Equation (8.2) is generally called the state equation(s), where lower-case boldface represents vectors and upper-case boldface represents matrices. Thus

x is the n dimensional state vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{8.3}$$

u is the m dimensional input vector

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \tag{8.4}$$

A is the $n \times n$ system matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \tag{8.5}$$

B is the $n \times m$ control matrix

$$\begin{bmatrix} b_{11} & \ldots & b_{1m} \\ b_{21} & \ldots & b_{2m} \\ \vdots & & \\ b_{n1} & \ldots & b_{nm} \end{bmatrix} \tag{8.6}$$

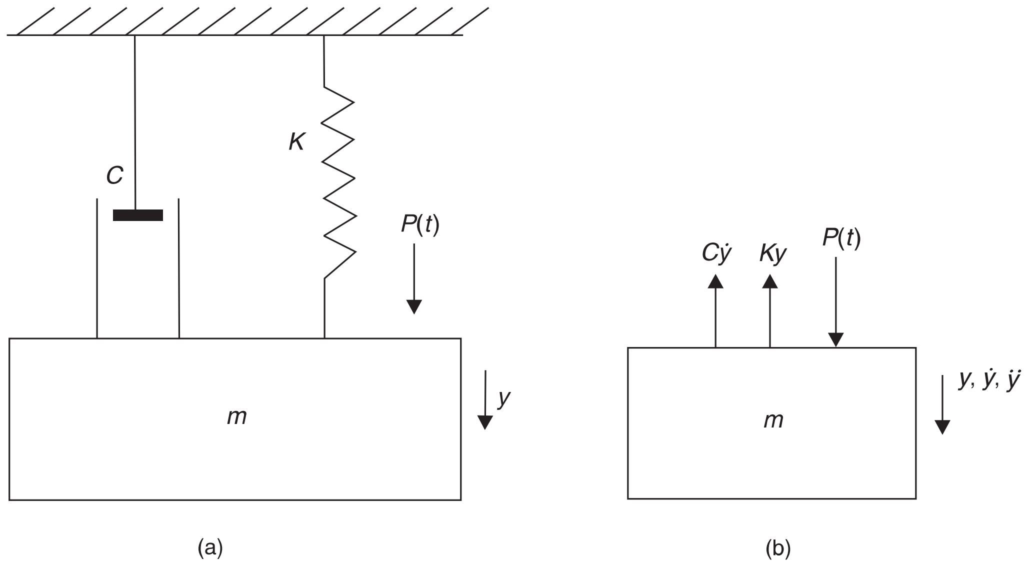

Fig. 8.1 Spring–mass–damper system and free-body diagram.

In general, the outputs $(y_1, y_2, \ldots, y_n)$ of a linear system can be related to the state variables and the input variables

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{8.7}$$

Equation (8.7) is called the output equation(s).

Example 8.1
Write down the state equation and output equation for the spring–mass–damper system shown in Figure 8.1(a).

Solution
State variables

$$x_1 = y \tag{8.8}$$

$$x_2 = \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{x}_1 \tag{8.9}$$

Input variable

$$u = P(t) \tag{8.10}$$

Now

$$\sum F_y = m\ddot{y}$$

From Figure 8.1(b)

$$P(t) - Ky - C\dot{y} = m\ddot{y}$$

or

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{K}{m}y - \frac{C}{m}\dot{y} + \frac{1}{m}P(t) \tag{8.11}$$

From equations (8.9), (8.10) and (8.11) the set of first-order differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{m}x_1 - \frac{C}{m}x_2 + \frac{1}{m}u \end{aligned} \tag{8.12}$$

and the state equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{m} & -\dfrac{C}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{8.13}$$

From equation (8.8) the output equation is

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{8.14}$$

State variables are not unique, and may be selected to suit the problem being studied.

Example 8.2
For the *RCL* network shown in Figure 8.2, write down the state equations when

(a) the state variables are $v_2(t)$ and $\dot{v}_2$
(b) the state variables are $v_2(t)$ and $i(t)$.

Solution
(a)

$$\begin{aligned} x_1 &= v_2(t) \\ x_2 &= \dot{v}_2 = \dot{x}_1 \end{aligned} \tag{8.15}$$

From equation (2.37)

$$LC\frac{d^2 v_2}{dt^2} + RC\frac{dv_2}{dt} + v_2 = v_1(t) \tag{8.16}$$

From equations (8.15) and (8.16) the set of first-order differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ x_2 &= -\frac{1}{LC}x_1 - \frac{RC}{LC}x_2 + \frac{1}{LC}u \end{aligned} \tag{8.17}$$

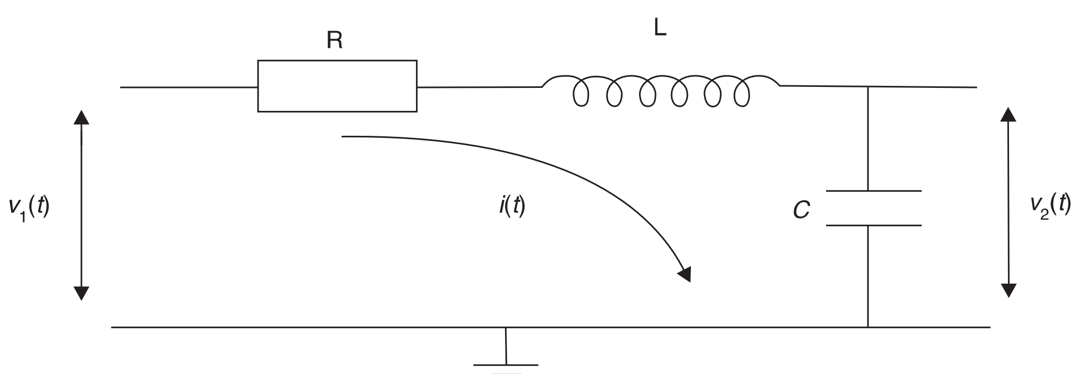

Fig. 8.2 *RCL* network.

and the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & \dfrac{-R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u \tag{8.18}$$

(b)

$$\begin{aligned} x_1 &= v_2(t) \\ x_2 &= i(t) \end{aligned} \tag{8.19}$$

From equations (2.34) and (2.35)

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -v_2(t) - Ri(t) + v_1(t) \tag{8.20}$$

$$C\frac{\mathrm{d}v_2}{\mathrm{d}t} = i(t) \tag{8.21}$$

Equations (8.20) and (8.21) are both first-order differential equations, and can be written in the form

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}x_2 \\ \dot{x}_2 &= -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}u \end{aligned} \tag{8.22}$$

giving the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} u \tag{8.23}$$

Example 8.3
For the 2 mass system shown in Figure 8.3, find the state and output equation when the state variables are the position and velocity of each mass.

Solution
State variables

$$\begin{aligned} x_1 &= y_1 \quad x_2 = \dot{y}_1 \\ x_3 &= y_2 \quad x_4 = \dot{y}_2 \end{aligned}$$

System outputs

$$y_1, y_2$$

System inputs

$$u = P(t) \tag{8.24}$$

For mass m_1

$$\sum F_y = m_1\ddot{y}_1$$

$$K_2(y_2 - y_1) - K_1y_1 + P(t) - C_1\dot{y}_1 = m_1\ddot{y}_1 \tag{8.25}$$

For mass m_2

$$\sum F_y = m_2\ddot{y}_2$$

$$-K_2(y_2 - y_1) = m_2\ddot{y}_2 \tag{8.26}$$

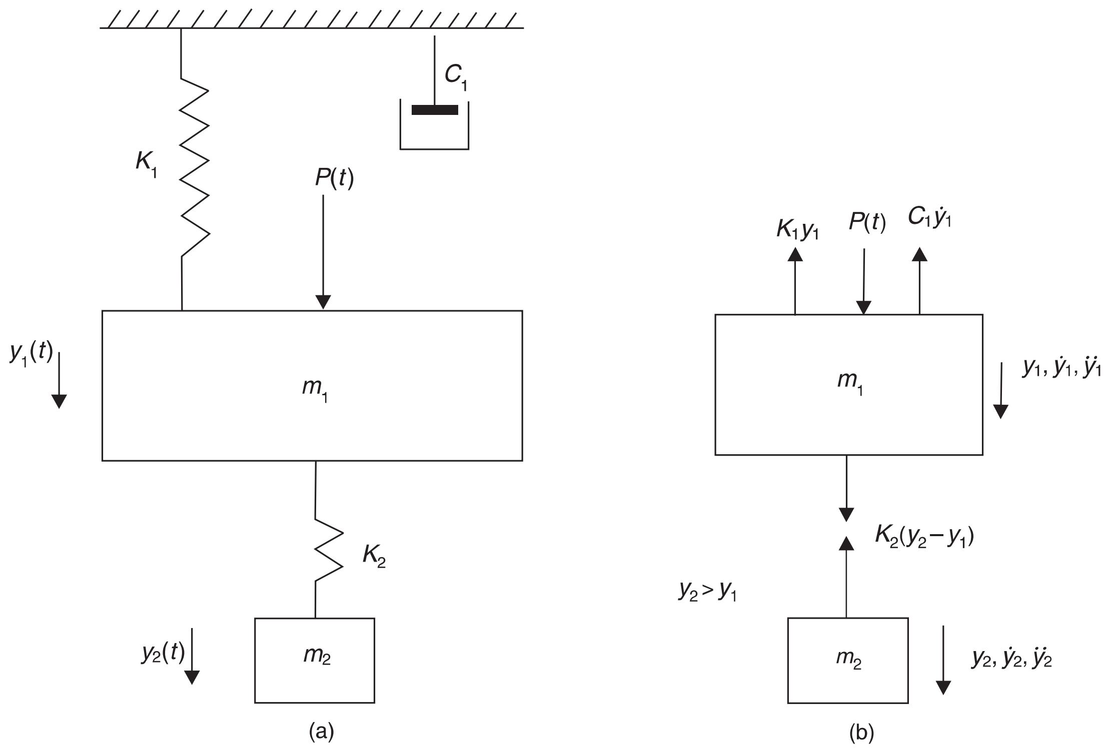

Fig. 8.3 Two-mass system and free-body diagrams.

From (8.24), (8.25) and (8.26), the four first-order differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \left(-\frac{K_1}{m_1} - \frac{K_2}{m_1}\right)x_1 - \frac{C_1}{m_1}x_2 + \frac{K_2}{m_1}x_3 + \frac{1}{m_1}u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{K_2}{m_2}x_1 - \frac{K_2}{m_2}x_3 \end{aligned} \tag{8.27}$$

Hence the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\dfrac{K_1 + K_2}{m_1}\right) & -\dfrac{C}{m_1} & \dfrac{K_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{m_2} & 0 & -\dfrac{K_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u \tag{8.28}$$

and the output equations are

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{8.29}$$

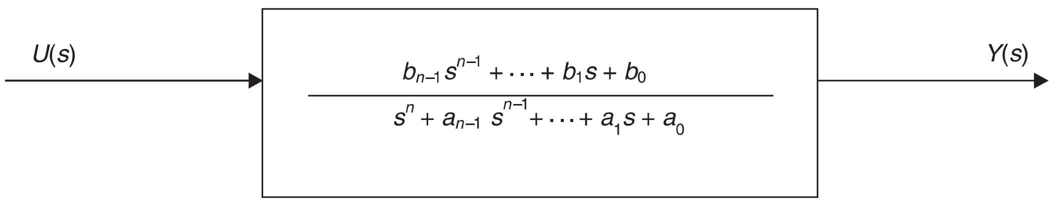

Fig. 8.4 Generalized transfer function.

8.1.3 State equations from transfer functions

Consider the general differential equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_{n-1}\frac{\mathrm{d}^{n-1} u}{\mathrm{d}t^{n-1}} + \cdots + b_1\frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u \tag{8.30}$$

Equation (8.30) can be represented by the transfer function shown in Figure 8.4.

Define a set of state variables such that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots \quad & \quad \vdots \\ \dot{x}_n &= -a_0x_1 - a_1x_2 - \cdots - a_{n-1}x_n + u \end{aligned} \tag{8.31}$$

and an output equation

$$y = b_0x_1 + b_1x_2 + \cdots + b_{n-1}x_n \tag{8.32}$$

Then the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{8.33}$$

The state-space representation in equation (8.33) is called the controllable canonical form and the output equation is

$$y = [\,b_0 \quad b_1 \quad b_2 \quad \dots \quad b_{n-1}\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \tag{8.34}$$

Example 8.4 (See also Appendix 1, *examp84.m*)
Find the state and output equations for

$$\frac{Y}{U}(s) = \frac{4}{s^3 + 3s^2 + 6s + 2}$$

Solution
State equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{8.35}$$

Output equation

$$y = [4 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.36}$$

Example 8.5
Find the state and output equations for

$$\frac{Y}{U}(s) = \frac{5s^2 + 7s + 4}{s^3 + 3s^2 + 6s + 2}$$

Solution
The state equation is the same as (8.35). The output equation is

$$y = [4 \quad 7 \quad 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{8.37}$$

8.2 Solution of the state vector differential equation

Consider the first-order differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax(t) + bu(t) \tag{8.38}$$

where $x(t)$ and $u(t)$ are scalar functions of time. Take Laplace transforms

$$sX(s) - x(0) = aX(s) + bU(s) \tag{8.39}$$

where $x(0)$ is the initial condition. From equation (8.39)

$$X(s) = \frac{x(0)}{(s-a)} + \frac{b}{(s-a)}U(s) \tag{8.40}$$

Inverse transform

$$x(t) = \mathrm{e}^{at}x(0) + \int_0^t \mathrm{e}^{a(t-\tau)}bu(\tau)\mathrm{d}\tau \tag{8.41}$$

where the integral term in equation (8.41) is the convolution integral and τ is a dummy time variable. Note that

$$\mathrm{e}^{at} = 1 + at + \frac{a^2t^2}{2!} + \cdots + \frac{a^kt^k}{k!} \tag{8.42}$$

Consider now the state vector differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.43}$$

Taking Laplace transforms

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{8.44}$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

Pre-multiplying by $(s\mathbf{I} - \mathbf{A})^{-1}$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{8.45}$$

Inverse transform

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{U}(\tau)\mathrm{d}\tau \tag{8.46}$$

if the initial time is t_0, then

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(0) + \int_{t_0}^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{8.47}$$

The exponential matrix $\mathrm{e}^{\mathbf{A}t}$ in equation (8.46) is called the state-transition matrix $\mathbf{\Phi}(t)$ and represents the natural response of the system. Hence

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \tag{8.48}$$

$$\mathbf{\Phi}(t) = \mathscr{L}^{-1}(s\mathbf{I} - \mathbf{A})^{-1} = \mathrm{e}^{\mathbf{A}t} \tag{8.49}$$

Alternatively

$$\mathbf{\Phi}(t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^k}{k!} \tag{8.50}$$

Hence equation (8.46) can be written

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \tag{8.51}$$

In equation (8.51) the first term represents the response to a set of initial conditions, whilst the integral term represents the response to a forcing function.

Characteristic equation

Using a state variable representation of a system, the characteristic equation is given by

$$|(s\mathbf{I} - \mathbf{A})| = 0 \tag{8.52}$$

8.2.1 Transient solution from a set of initial conditions

Example 8.6
For the spring–mass–damper system given in Example 8.1, Figure 8.1, the state equations are shown in equation (8.13)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{K}{m} & -\dfrac{C}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{8.53}$$

Given: $m = 1\,\text{kg}$, $C = 3\,\text{Ns/m}$, $K = 2\,\text{N/m}$, $u(t) = 0$. Evaluate,

(a) the characteristic equation, its roots, ω_n and ζ
(b) the transition matrices $\phi(s)$ and $\phi(t)$
(c) the transient response of the state variables from the set of initial conditions

$$y(0) = 1.0,$$
$$\dot{y}(0) = 0$$

Solution
Since $x_1 = y$ and $x_2 = \dot{y}$, then $x_1(0) = 1.0$, $x_2(0) = 0$.
Inserting values of system parameters into equation (8.53) gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(a)
$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & (s+3) \end{bmatrix} \tag{8.54}$$

From equation (8.52), the characteristic equation is

$$|(s\mathbf{I} - \mathbf{A})| = s(s+3) - (-2) = s^2 + 3s + 2 = 0 \tag{8.55}$$

Roots of characteristic equation

$$s = -1, -2 \tag{8.56}$$

Compare equation (8.55) with the denominator of the standard form in equation (3.43)

$$\begin{aligned} \omega_n^2 &= 2 \quad \text{i.e} \quad \omega_n = 1.414\,\text{rad/s} \\ 2\zeta\omega_n &= 3 \quad \text{i.e} \quad \zeta = 1.061 \end{aligned} \tag{8.57}$$

(b) The inverse of any matrix $\mathbf{A}$ (see equation A2.17) is

$$\mathbf{A}^{-1} = \frac{\text{Adjoint } \mathbf{A}}{\det \mathbf{A}} \tag{8.58}$$

From equation (8.48)

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

Using the standard matrix operations given in Appendix 2, equation (A2.12)

$$\text{Minors of } \boldsymbol{\Phi}(s) = \begin{bmatrix} (s+3) & 2 \\ -1 & s \end{bmatrix}$$

$$\text{Co-factors of } \boldsymbol{\Phi}(s) = \begin{bmatrix} (s+3) & -2 \\ 1 & s \end{bmatrix}$$

The Adjoint matrix is the transpose of the Co-factor matrix

$$\text{Adjoint of } \boldsymbol{\Phi}(s) = \begin{bmatrix} (s+3) & 1 \\ -2 & s \end{bmatrix} \tag{8.59}$$

Hence, from equations (8.58) and (8.48)

$$\boldsymbol{\Phi}(s) = \begin{bmatrix} \dfrac{(s+3)}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} \tag{8.60}$$

Using partial fraction expansions

$$\boldsymbol{\Phi}(s) = \begin{bmatrix} \left(\dfrac{2}{s+1} - \dfrac{1}{s+2}\right) & \left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) \\ -2\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) & \left(-\dfrac{1}{s+1} + \dfrac{2}{s+2}\right) \end{bmatrix} \tag{8.61}$$

Inverse transform equation (8.61)

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}) \end{bmatrix} \tag{8.62}$$

Note that the exponential indices are the roots of the characteristic equation (8.56).

(c) From equation (8.51), the transient response is given by

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{x}(0) \tag{8.63}$$

Hence

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) & (-\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8.64}$$

$$\begin{aligned} x_1(t) &= (2\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \\ x_2(t) &= -2(\mathrm{e}^{-t} - \mathrm{e}^{-2t}) \end{aligned} \tag{8.65}$$

The time response of the state variables (i.e. position and velocity) together with the state trajectory is given in Figure 8.5.

Example 8.7
For the spring–mass–damper system given in Example 8.6, evaluate the transient response of the state variables to a unit step input using

(a) The convolution integral
(b) Inverse Laplace transforms

Assume zero initial conditions.

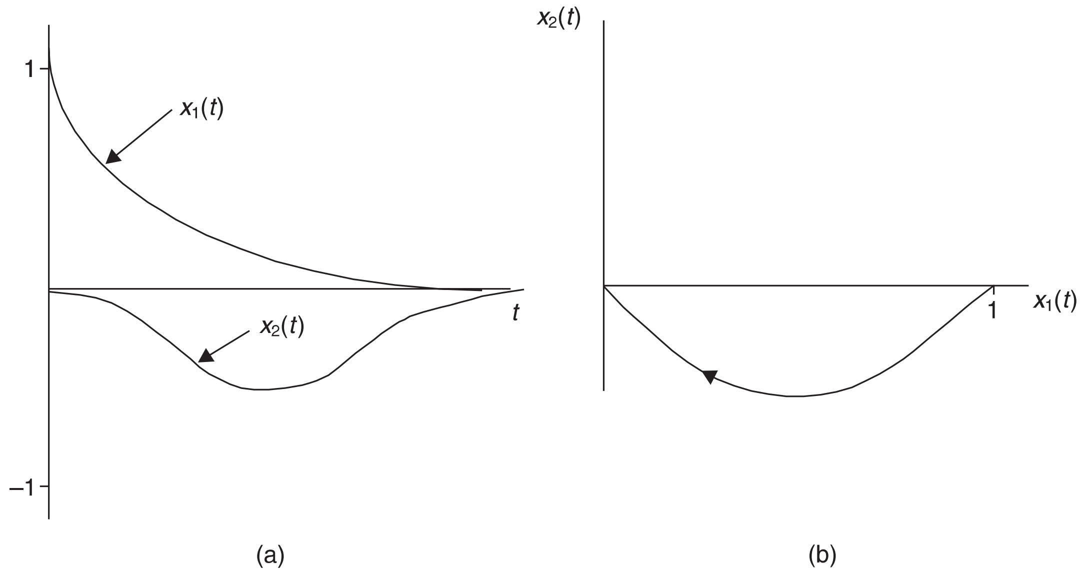

Fig. 8.5 State variable time response and state trajectory for Example 8.4.

Solution

(a) From equation (8.51)

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} \mathbf{u}(\tau)\mathrm{d}\tau \tag{8.66}$$

Given that $u(t) = 1$ and $1/m = 1$, equation (8.66) reduces to

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} \phi_{12}(t-\tau) \\ \phi_{22}(t-\tau) \end{bmatrix} \mathrm{d}\tau$$

Inserting values from equation (8.62)

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} \mathrm{e}^{-(t-\tau)} - \mathrm{e}^{-2(t-\tau)} \\ \mathrm{e}^{-(t-\tau)} + 2\mathrm{e}^{-2(t-\tau)} \end{bmatrix} \mathrm{d}\tau \tag{8.67}$$

Integrating

$$\mathbf{x}(t) = \begin{bmatrix} \mathrm{e}^{-(t-\tau)} - \frac{1}{2}\mathrm{e}^{-2(t-\tau)} \\ \mathrm{e}^{-(t-\tau)} + \mathrm{e}^{-2(t-\tau)} \end{bmatrix}_0^t \tag{8.68}$$

Inserting integration limits ($\tau = t$ and $\tau = 0$)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\mathrm{e}^{-2t} \\ \mathrm{e}^{-t} - \mathrm{e}^{-2t} \end{bmatrix} \tag{8.69}$$

(b) An alternative method is to inverse transform from an s-domain expression. Equation (8.45) may be written

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{BU}(s) \tag{8.70}$$

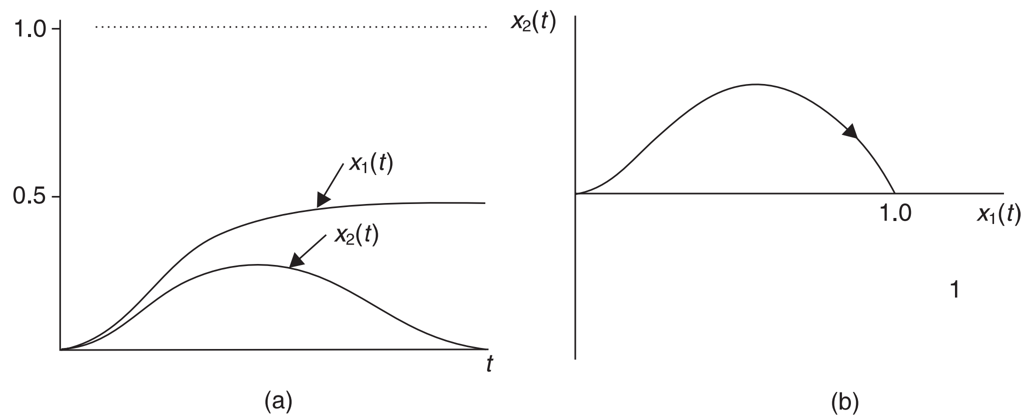

Fig. 8.6 State variable step response and state trajectory for Example 8.5.

Hence from equation (8.61)

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \left(\frac{2}{s+1} - \frac{1}{s+2}\right) & \left(\frac{1}{s+1} - \frac{1}{s+2}\right) \\ -2\left(\frac{1}{s+1} - \frac{1}{s+2}\right) & \left(\frac{-1}{s+1} + \frac{2}{s+2}\right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \tag{8.71}$$

Simplifying

$$\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s(s+1)} - \frac{1}{2}\left\{\frac{2}{s(s+2)}\right\} \\ \frac{-1}{s(s+1)} + \frac{2}{s(s+2)} \end{bmatrix} \tag{8.72}$$

Inverse transform

$$\mathbf{x}(t) = \begin{bmatrix} (1 - e^{-t}) - \frac{1}{2}(1 - e^{-2t}) \\ -(1 - e^{-t}) + (1 - e^{-2t}) \end{bmatrix} \tag{8.73}$$

which gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \tag{8.74}$$

Equation (8.74) is the same as equation (8.69).

The step response of the state variables, together with the state trajectory, is shown in Figure 8.6.

8.3 Discrete-time solution of the state vector differential equation

The discrete-time solution of the state equation may be considered to be the vector equivalent of the scalar difference equation method developed from a z-transform approach in Chapter 7.