# Time - Domain Analysis of control system

The time response of a control system consists of two parts : the transient response and the steady-state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as (t) approaches infinity. Thus the system response c(t) may be written as

$$c(t) = C_{tr}(t) + C_{ss}(t)$$

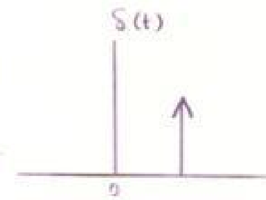
where

$C_{tr}(t)$ .... transient response.

$C_{ss}(t)$ .... steady-state response.

The typical test signals for the time response of control system are :

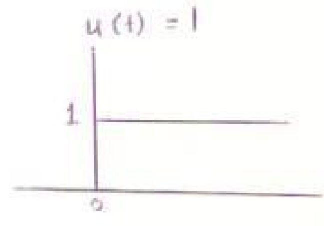
① - Unit Impulse Response : $\delta(s) = 1$




the time response is the inverse Laplace transform of G(s)

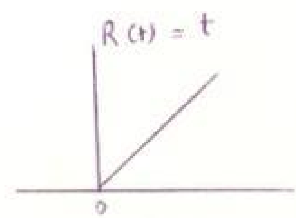
$$y(t) = \mathcal{L}^{-1} G(s)$$

②. Unit step input : $U(s) = \frac{1}{s}$

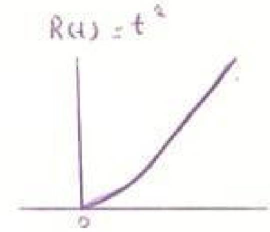



$$y(t) = \mathcal{L}^{-1} G(s) \cdot U(s)$$

③ - Ramp input : $R(s) = \frac{1}{s^2}$




$$y(t) = \mathcal{L}^{-1} G(s) \cdot R(s)$$



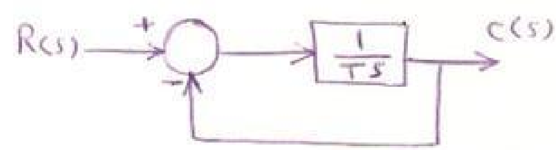

④. Parabolic input : $R(s) = \frac{1}{s^3}$

$$y(t) = \mathcal{L}^{-1}\, G(s) \cdot R(s)$$

## Transient Response

### ①. Transient Response of first order system :-

$$\frac{C(s)}{R(s)} = \frac{1}{1+TS}$$




First order system

(a) Unit-step Response : $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{TS+1} \cdot \frac{1}{S}$$

By using partial fraction gives

$$C(s) = \frac{1}{S} - \frac{T}{TS+1} = \frac{1}{S} - \frac{1}{S+(\frac{1}{T})}$$

Taking the inverse Laplace transform gives

$$c(t) = 1 - e^{-(t/T)} \quad \text{for } t \geq 0$$

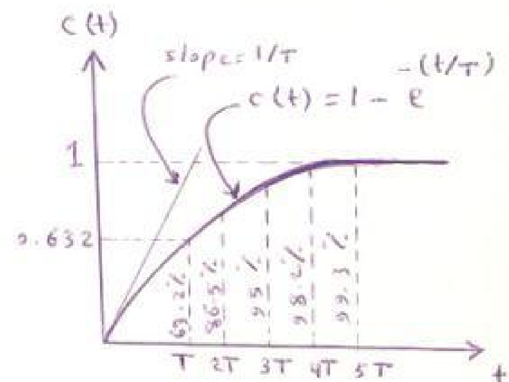



Fig. Exponential response curve

Notes :

1. $c(t)$ initially zero and finally one.
2. At $t = T$, the value of $c(t)$ is $0.632$ or $c(t)$ reach 63.2% of its final value, where $T$ is time constant of the system.
3. The exponential response curve is that the slope of the tangent line at $t=0$ is $1/T$.

$$\left.\frac{dc}{dt}\right|_{t=0} = \left.\frac{1}{T} e^{-t/T}\right|_{t=0} = 1/T$$

(b) Unit-Ramp Response : $R(s) = \frac{1}{s^2}$

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2}$$

by partial fraction gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

taking inverse Laplace transform, we obtain

$$c(t) = t - T + T e^{-t/T}, \quad \text{for } t \geq 0$$

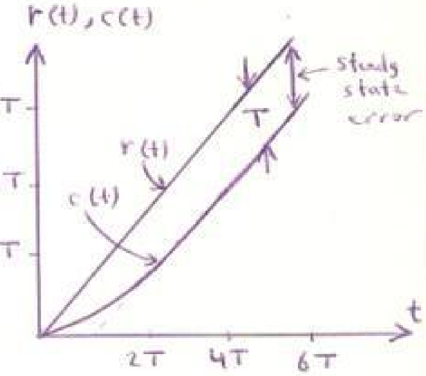



Fig. Unit-ramp response

The error signal $e(t)$ is then

$$e(t) = r(t) - c(t) = T(1 - e^{-t/T})$$

$$e(\infty) = T$$

(c) Unit-Impulse Response : $R(s) = 1$

$$C(s) = \frac{1}{Ts+1}$$

The inverse Laplace transform gives

$$c(t) = \frac{1}{T} e^{-t/T}, \quad \text{for } t \geq 0$$

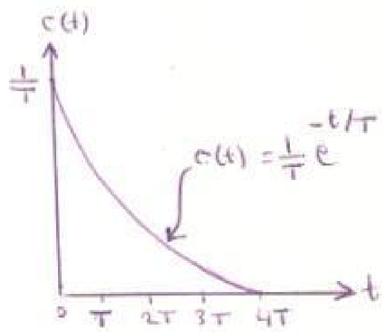



Fig. Unit-impulse response

② - Transient Response of second order system:-

For Servo system

$$J\ddot{c} + B\dot{c} = T$$

$$J s^2 c(s) + B s\, c(s) = T(s)$$

$$\frac{C(s)}{T(s)} = \frac{1}{S(JS+B)}$$

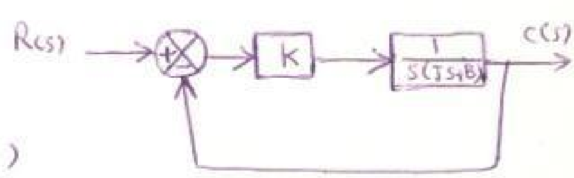



Fig. Servo system

The closed-Loop transfer function is then obtained as

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2+Bs+K} = \frac{(K/J)}{s^2+(B/J)s+(K/J)}$$

This T.F can be rewritten as

$$\frac{C(s)}{R(s)} = \frac{K/J}{\left[s+\frac{B}{2J}+\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]\left[s+\frac{B}{2J}-\sqrt{\left(\frac{B}{2J}\right)^2-\frac{K}{J}}\right]}$$

The closed-loop poles are complex conjugates if $B^2-4JK<0$ and they are real if $B^2-4JK\geq 0$.

In transient response analysis, it is convenient to write

$$\frac{K}{J} = \omega_n^2 \quad , \quad \frac{B}{J} = 2\zeta\omega_n = 2\sigma$$

where $\sigma$ -- is called the attenuation

$\omega_n$ -- the undamped natural frequency

$\zeta$ -- the damping ratio.

The damping ratio ($\zeta$) is the ratio of the actual damping (B) to the critical damping $B_c = 2\sqrt{JK}$

$$\zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

hence the T.F can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

This form is called the standard form of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of two parameters $\zeta$ and $\omega_n$. If $0 < \zeta < 1$, the closed-Loop poles are complex conjugates and Lie in the Left-half S-plane. The system is then called under damped, and the transient response is oscillatory. If $\zeta = 0$, the transient response does not die out. If $\zeta = 1$, the system is called critically damped. Over damped systems corresponed to $\zeta > 1$.

We shall now solve for the response of the second order system (servo system) to a unit-step input for three cases: the underdamped $(0 < \zeta < 1)$, critically $(\zeta = 1)$, and overdamped $(\zeta > 1)$.

**(a) Underdamped case $(0 < \zeta < 1)$:**

In this case, $\frac{C(s)}{R(s)}$ can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The frequency $\omega_d$ is called damped natural frequency.

For a unit-step input, $C(t)$ can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

Inverse Laplace transform:

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t}\sin\omega_d t$$

hence, $\mathcal{L}^{-1}[C(s)] = c(t)$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \text{ for } t \geq 0$$

$$e(t) = r(t) - c(t)$$

$$= e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right), \text{ for } t \geq 0$$

If $\zeta = 0$,

$$c(t) = 1 - \cos\omega_n t, \text{ for } t \geq 0$$

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## (b) Critically damped case ($\zeta = 1$):

If the two poles of $\frac{C(s)}{R(s)}$ are equal, the system is said to be a critically damped one.

For a unit-step input, $R(s) = \frac{1}{s}$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s}$$

Taking inverse Laplace transform gives

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t), \quad \text{for } t \geq 0$$

## (c) Overdamped case ($\zeta > 1$):

In this case, the two poles of $\frac{C(s)}{R(s)}$ are negative real and unequal.

For a unit-step input, $R(s) = \frac{1}{s}$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})\, s}$$

The inverse Laplace transform for above T.F is

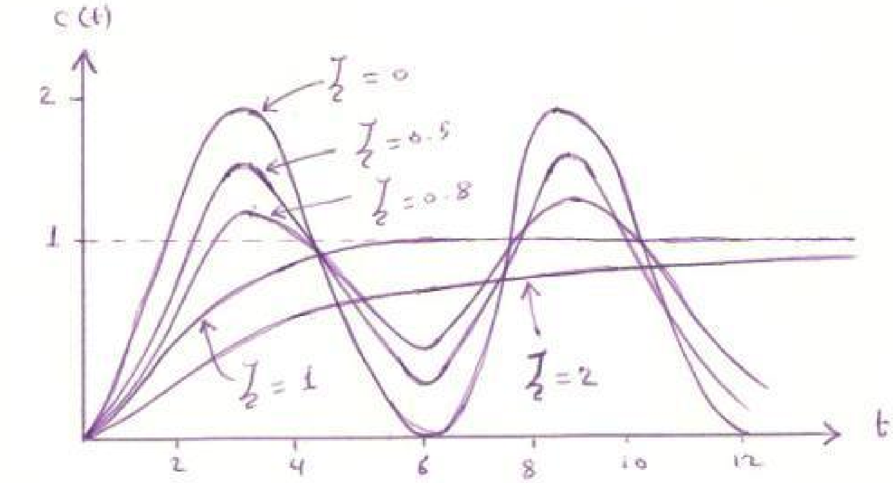
$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})}\, e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t}$$

$$- \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})}\, e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$= 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right), \quad \text{for } t \geq 0$$

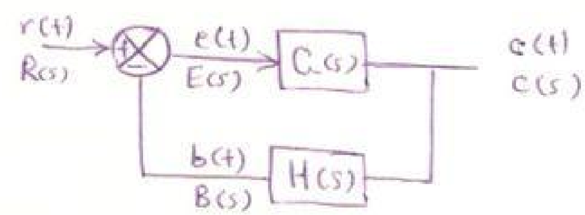
Where $s_1 = (\zeta+\sqrt{\zeta^2-1})\omega_n$ and $s_2 = (\zeta-\sqrt{\zeta^2-1})\omega_n$

Thus, the response $c(t)$ includes two decaying exponential terms.




## Steady - state Error

The steady-state error is a measure of accuracy and in designed problem one of the objected is to keep this error to minimum value.




$$e(t) = r(t) - b(t)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)\,C(s)$$
$$= R(s) - H(s)\,G(s)\,E(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

steady-state error $= e_{ss} = \lim_{t \to \infty} e(t)$

using final value theorem, $e_{ss}$ is

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)H(s)}$$