Types of control system :

In general $G(s)\,H(s)$ may be written as

$$G(s)\,H(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots\cdots(1+T_m s)}{s^J (1+T_a s)(1+T_b s)\cdots\cdots(1+T_n s)}$$

Where K, Ts -- are constants

J ---- type of the system

$J = 0 \Rightarrow$ type 0

$J = 1 \Rightarrow$ type 1

$J = 2 \Rightarrow$ type 2

Ex:

$$G(s)\,H(s) = \frac{K(s+1)}{(s+2)(s+6)} \Rightarrow \text{type } 0$$

$$G(s)\,H(s) = \frac{K}{(s^2+2s)} \Rightarrow \text{type } 1$$

static Position Error Constant K_p :

The steady-state error of the system for a unit-step input is :

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)\,H(s)} \cdot \frac{1}{s} = \frac{1}{1+\lim\limits_{s\to 0} G(s)\,H(s)}$$

The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s)\,H(s)$$

$$\therefore e_{ss} = \frac{1}{1+K_p}$$

for type 0 system

$$K_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots} = K$$

For a type 1 or higher system

$$k_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)\cdots}{s^J(1+T_a s)(1+T_b s)\cdots} = \infty \quad , \text{for } J \geq 1$$

$$\therefore \quad e_{ss} = \frac{1}{1+K} \quad \text{for typ 0 systems}$$

$$e_{ss} = 0 \quad \text{for type 1 or higher systems}$$

static velocity Error Constant K_v :

The steady-state error of the system with a unit ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{s+sG(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{sG(s)H(s)}$$

The static velocity error constant K_v is defined by

$$K_v = \lim_{s\to 0} s\,G(s)\,H(s)$$

$$\therefore \quad e_{ss} = \frac{1}{K_v}$$

For a type 0 system, $K_v = \lim_{s\to 0} \frac{sK(1+T_1 s)(1+T_2 s)\cdots}{(1+T_a s)(1+T_b s)\cdots} = 0$

For a type 1 system,

$$K_v = \lim_{s\to 0} \frac{sK(1+T_1 s)(1+T_2 s)\cdots}{s(1+T_a s)(1+T_b s)\cdots} = K$$

For a type 2 or higher system,

$$K_v = \lim_{s\to 0} \frac{sK(1+T_1 s)(1+T_2 s)\cdots}{s^J(1+T_a s)(1+T_b s)\cdots} = \infty \quad , \text{for } J \geq 2$$

$$\therefore \quad e_{ss} = \frac{1}{K_v} = \infty \quad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \quad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0 \quad, \text{for type 2 or higher systems}$$

Static Acceleration Error Constant K_a :

The steady - state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \cdot \frac{1}{s^3} = \frac{1}{\lim_{s \to 0} s^2 G(s) H(s)}$$

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$e_{ss} = \frac{1}{K_a}$$

For a type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(1+T_1 s)(1+T_2 s) \cdots}{(1+T_a s)(1+T_b s) \cdots} = 0$$

For a type 1 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(1+T_1 s)(1+T_2 s) \cdots}{s(1+T_a s)(1+T_b s) \cdots} = 0$$

For a type 2 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(1+T_1 s)(1+T_2 s) \cdots}{s^2(1+T_a s)(1+T_b s) \cdots} = K$$

For a type 3 or higher system

$$K_a = \lim_{s \to 0} \frac{s^2 K(1+T_1 s)(1+T_2 s) \cdots}{s^J (1+T_a s)(1+T_b s) \cdots} = \infty, \quad \text{for } J \geqslant 3$$

$\therefore\ e_{ss} = \infty$, for type 0 and type 1 systems

$e_{ss} = \frac{1}{K}$, for type 2 systems

$e_{ss} = 0$, for type 3 or higher systems

type	K_p	K_v	K_a	e_{ss} step	e_{ss} Ramp	e_{ss} Parabolic
0	K	0	0	$\frac{1}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{1}{K}$	∞
2	∞	∞	K	0	0	$\frac{1}{K}$
3	∞	∞	∞	0	0	0

Table – K_p, K_v, K_a, and steady-state error in terms of Gain K.

Definitions of Transient - Response Specifications

① - Delay time t_d : The delay time is the time required for the response to reach half the final value the very first time.

② - Rise time t_r : The rise time is the time required for the response to rise from 10% to 90% of its final value.

③ - Peak time t_p : The peak time is the time required for the response to reach the first peak of the overshot.

④ - Maximum (percent) overshot M_p : The maximum overshot is the maximum peak value of the response curve measured from unity.

$$\text{Maximum percent overshot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

⑤ - Settling time t_s : The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).

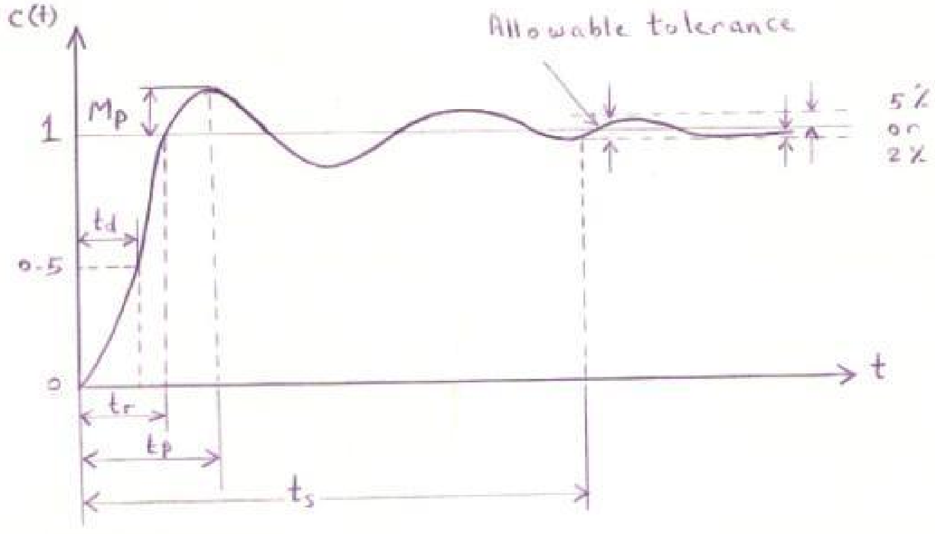


Fig. Unit-step response curve showing t_d, t_r, t_p, M_p, and t_s.

In the following, we shall obtain the rise time, peak time, maximum overshot, and settling time of the second-order system given by equation:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The system is assumed to be underdamped.

<u>Rise time tr:</u> Referring to equation:

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

We obtain the rise time t_r by letting $c(t_r) = 1$

$$c(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r\right)$$

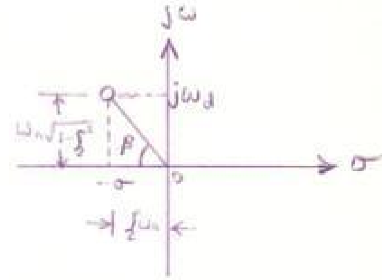
Since $e^{-\zeta\omega_n t_r} \neq 0$, we obtain

$$\cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

or $\tan \omega_d t_r = -\dfrac{\sqrt{1-\zeta^2}}{\zeta} = -\dfrac{\omega_d}{\sigma}$

$\therefore \quad t_r = \dfrac{1}{\omega_d} \tan^{-1}\left(\dfrac{\omega_d}{-\sigma}\right)$

$t_r = \dfrac{\pi - \beta}{\omega_d}$



Peak time t_p : Referring to equation :

$$c(t) = 1 - e^{-\zeta \omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right)$$

$$\frac{dc(t)}{dt} = \zeta \omega_n e^{-\zeta \omega_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right) +$$

$$+ e^{-\zeta \omega_n t}\left(\omega_d \sin \omega_d t - \frac{\zeta \omega_d}{\sqrt{1-\zeta^2}} \cos \omega_d t\right)$$

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p} = \left(\sin \omega_d t_p\right) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t_p} = 0$$

$\sin \omega_d t_p = 0$

or $\omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$

Since the Peak time corresponds to the first peak overshot :

$\omega_d t_p = \pi$

hence $t_p = \dfrac{\pi}{\omega_d}$

Maximum overshot M_p :

The maximum overshot occurs at the peak time or at $t = t_p = \frac{\pi}{\omega_d}$.

Assuming that the final value of the output is unity.

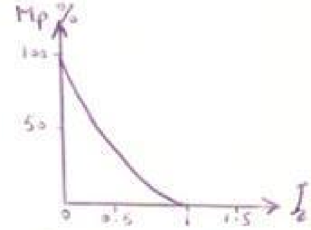
$$M_p = c(t_p) - 1 = -e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} = e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi}$$

The maximum percent overshot is $e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} \times 100\,\%$

If the final value $c(\infty)$ of the output is not unity, then

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

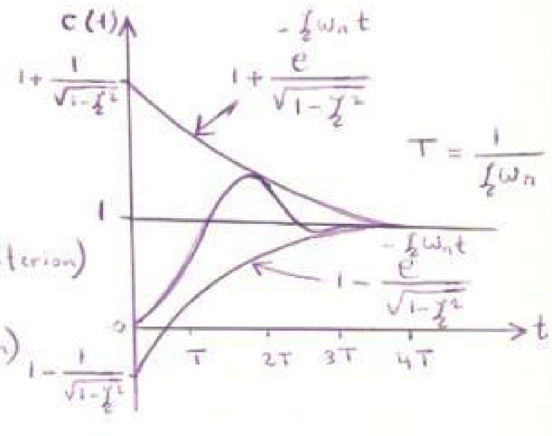


Settling time t_s :

For an undamped second-order system, the transient response is obtained from equation:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right), \quad \text{for } t \geq 0$$

The curves $1 \mp \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)$ are the envelope curves of the transient response to a unit-step input.

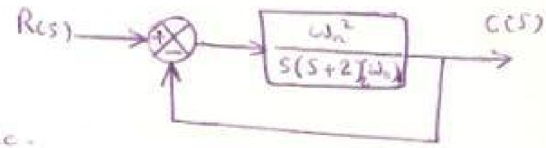


$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (2\% \text{ criterion})$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (5\% \text{ criterion})$$

IV

Ex: Consider the system shown in figure, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec.



Find t_r, t_p, M_p, and t_s when the system is subjected to a unit-step input.

Solution:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-(0.6)^2} = 4$$

$$\sigma = \omega_n \zeta = (5)(0.6) = 3$$

① $$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

$$\therefore\ t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ Sec}$$

② $$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ Sec}$$

③ $$M_p = e^{-\left(\frac{\sigma}{\omega_d}\right)\pi} = e^{-\left(\frac{3}{4}\right)*3.14} = 0.095$$

The maximum percent over shot is thus 9.5 %

④ For the 2% criterion

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ Sec}$$

For the 5 % criterion

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$