Stability of Control System

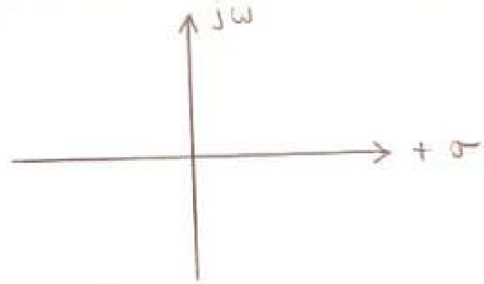
Complex plane

The complex quantity

$$S = \sigma + j\omega$$

where σ and ω are real variables.

The plane in which the real axis is represented by (σ) and the imajenary axis is represented by (ω) is refered to as the complex plane or the s-plane.



Poles and zeros

Most T.F.s are expressed in terms of (S), as a ratio of two polynomials, i.e

$$T.F = \frac{(S-z_1)(S-z_2)\text{ ---}}{(S-P_1)(S-P_2)\text{ ---}} = F(s)$$

Each value of (s) which makes F(s) zero is called as a zero of F(s).

And each value of (s) which makes F(s) infinity is called a pole of F(s).

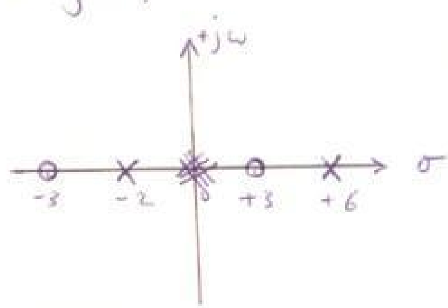
Poles are usually represented by (x).

Zeros are usually represented by (o).

Ex. Find the poles and zeros of $F(s)$.

where $F(s) = \dfrac{s^2 - 9}{s^3(s+2)(s-6)} = \dfrac{(s-3)(s+3)}{s^3(s+2)(s-6)}$

There are two zeros and five poles (three of which are at the origin).

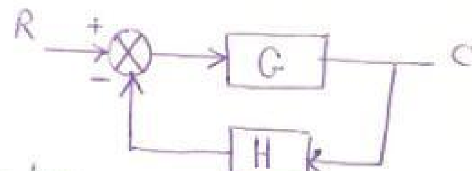


Characteristic Equation

It is the equation formed by puting the denominator of the T.F. of the system equal to zero.

i.e

$$C/L.T.F = \frac{G}{1+GH}$$



the C.E of this system is given by:

$$1 + GH = 0$$

If $T.F = \dfrac{a_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$

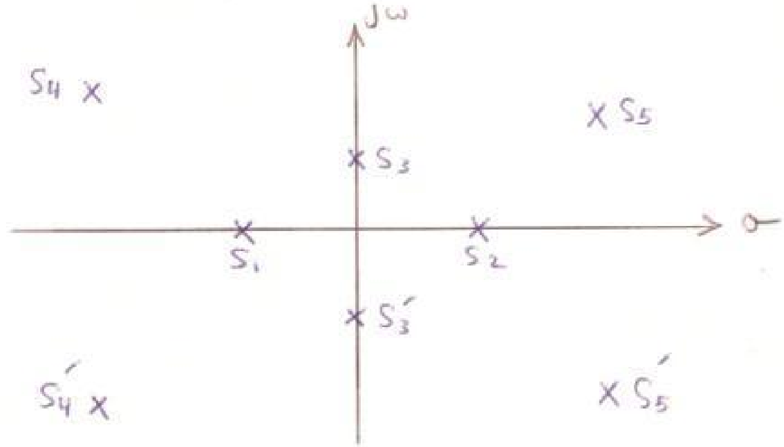
Then $s^n + a_{n-1}s^{n-1} + \cdots + a_0 = 0$ is the C.E.

The roots of this equation are the poles of the T.F

Definition of stability

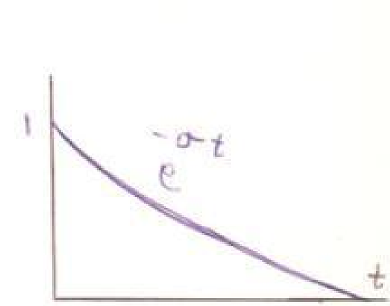
A system is said to be stable if for every bounded input, the output remains bounded.
Also, a system is stable if none of the poles of the closed-Loop T.F may Lie in the right hand half of the S-plane.

<u>Ex.</u> In the diagram shown below, explain with the aid of diagrams how the location of poles infleunes the system behaviour.



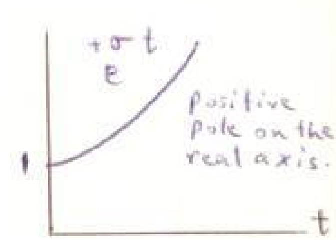
Ans. Using $S = \sigma + j\omega$

S_1 : implies a pole on the real axis where $\omega = 0$. This implies a simple exponential term in the solution.
Since S_1 is in the L.h.s of the S-plane
$\therefore$ σ must be negative.



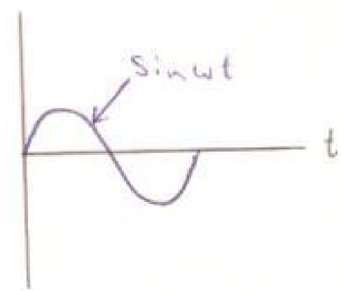
Therefore, the solution is a decaying exponential.

S_2 : is a pole on the real axis, but since it is in the r.h.s of the s-plane.
It therefore represents as exponential rice as σ is positive.

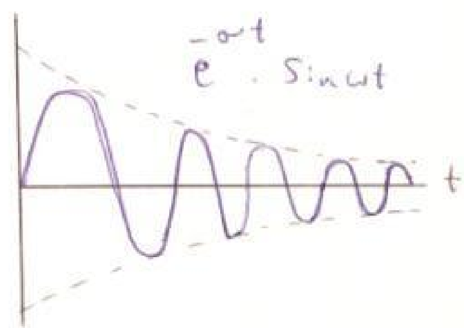


The poles S_3 and S_3' : imply an oscillatory term in the solution.

Because, they are on the $j\omega$ axis, there is no exponential term in the solution of these poles.

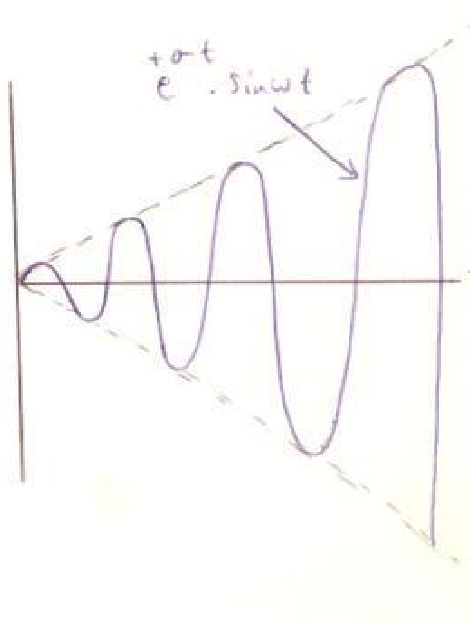


S_4 and S_4' : are Conjugate pair of poles off both axis.
Such poles on the L.h.s of the s-plane imply as oscillatory term which decays in magnitude exponentially.



The poles S_5 and S_5' :

are Conjugate pairs of poles off both axis, but in the r.h.s of the s-plane.
Therefore, the solution must be exponentially increasing oscillations.



The stability of control system divided in to:

①- Absolute stability: It refers to the condition of stable or unstable, it is a yes or no condition, if the system has one positive pole (positive exponential) it is enough to say that the system is unstable. Another factor may effect the stability that is the amplifier gain (K), all the system has negative pole, i.e suppose to be stable but it meet be unstable if the amplifier gain increas over certain value called (K_{cr}).

②. Relative stability:- Once the system is found to be stable it is of interest to determine how stable it is and this degree of stability is a measure of relative stability.

Methods of determining stability:

1- Routh - Hurwitz Criterion.
2- Root Locus plot.
3- Nyquist Criterion.
4. Bode diagram.
5- Lyapunov's stability criterion.

Routh's stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them.

This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, iformation about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

①- Write the polynomial in (s) in the following form:

$$a_n S^n + a_{n-1} S^{n-1} + a_{n-2} S^{n-2} + \cdots + a_0 = 0$$

where the coefficient are real quantities.

②- If any of the coefficient are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable.

③- If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$\begin{array}{c|cccc}
S^n & a_n & a_{n-2} & a_{n-4} & \cdots\cdots a_0 \\
S^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
S^{n-2} & b_1 & b_2 & b_3 & \cdots \\
S^{n-3} & C_1 & C_2 & C_3 & \cdots
\end{array}$$

$$b_1 = \frac{a_{n-1}\, a_{n-2} - a_n\, a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}\, a_{n-4} - a_n\, a_{n-5}}{a_{n-1}}$$

Then continue solve for b_3 and b_4 in the same way until obtain zeros.

$$c_1 = \frac{b_1\, a_{n-3} - b_2\, a_{n-1}}{b_1}$$

$$c_2 = \frac{b_1\, a_{n-5} - b_3\, a_{n-1}}{b_1}$$

Then continue to solve for C_3 and C_4 in the same way until obtain zeros after s^0.

If all the constants in the 1st column have the same sign (either +ve or −ve) then the system is stable. If the first column have +ve and −ve sign, then the system unstable. If one of these values equal to zero then the system is critical and the value of critical amplifier gain can be obtained accordingly.

<u>Ex1</u> Let us apply Routh's stability criterion to the following third order polynomial;

$$a_0 s^3 + a_1 s^2 + a_2 s + a_0 = 0$$

<u>Sol.</u>

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \frac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

$a_1 a_2 > a_3 a_0$ for the system to be stable.

Ex 2 Discuss stability using Routh criterion

$$s^3 - 2s^2 + 2s = 0$$

s^3	1	2	0
s^2	-2	0	0
s^1	2	0	0
s^0	0	0	0
	0	0	0

System unstable

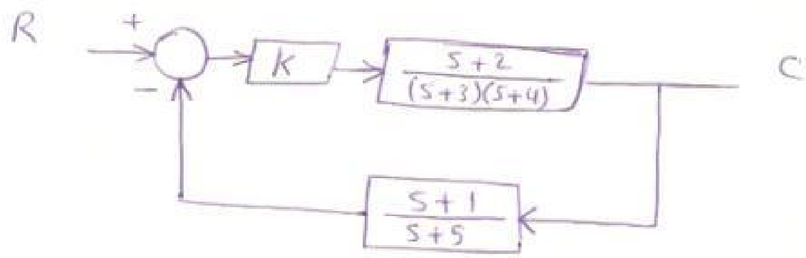
Ex 3 Discuss stability using Routh criterion

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

s^4	2	3	10	0
s^3	1	5	0	0
s^2	-7	10	0	0
s^1	63	0	0	0
s^0	10	0	0	0
	0	0	0	0

system unstable

H.W1 Find the value of K for the system to be stable.



H.W2 Discuss stability equation

$$s^6 + 2s^5 + 8s^4 + 13s^3 + 20s^2 + 16s + 16 = 0$$