Root - Locus Analysis

Root Locus : It is the Locus of roots of the characteristic equation of the closed-Loop system as a specific parameter (usually, gain K) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-Loop pole or zero to the Locations of the closed-Loop poles.

Angle and Magnitude Conditions :

consider the transfer function :

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

The c/c equation is : $1 + G(s)\,H(s) = 0$

or $G(s)\,H(s) = -1$

Since $G(s)\,H(s)$ is complex quantity then

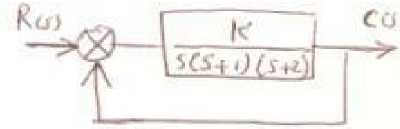
$G(s)\,H(s) = 1 \angle 180^\circ$

* Angle condition $\Rightarrow \angle G(s)\,H(s) = \mp 180(2k+1)$; $k = 0, 1, 2, \ldots$

* Magnitude Condition $\Rightarrow |G(s)\,H(s)| = 1$

A Locus of the points in the complex plane satisfying the angle condition alone is the root Locus. The roots of the c/c equation (the closed Loop poles) corresponding to a given value of the gain can be determined from the magnitude condition.

Ex: Consider the system shown in fig.



$$G(s) = \frac{K}{s(s+1)(s+2)} \quad , \quad H(s) = 1$$

The angle condition is :

$$\angle G(s) = \angle \frac{K}{s(s+1)(s+2)} = -\angle s - \angle s+1 - \angle s+2$$

$$= \mp 180\,(2n+1) \quad ; \quad (n = 0, 1, 2, \ldots)$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$$

Rules of Root Locus

①. The number of Loci of the plot is equal to the order of the characteristic equation.

②. Each Locus start at an open Loop pole when $(K = 0)$ and finishes either an open Loop zero or infinite $(K = \infty)$

③. Locate the poles and zeros of $G(s)\,H(s)$ on the s-plane

④. Loci either moves along the real axis or occurs at the complex conjugate pairs of Loci (real axis acts a mirror).

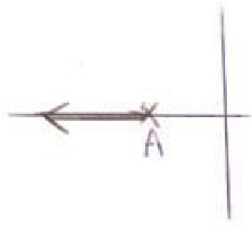
⑤. Determine the asymptotes of root Loci.

$$\text{Angles of asymptotes} = \frac{\mp 180\,(2n+1)}{P - Z} \quad (n = 0, 1, 2, \ldots)$$

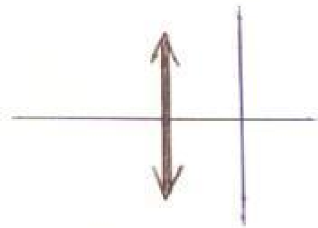
Where

P = number of poles of $G(s)\,H(s)$.

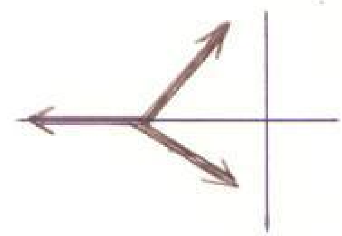
Z = number of zeros of $G(s)\,H(s)$.



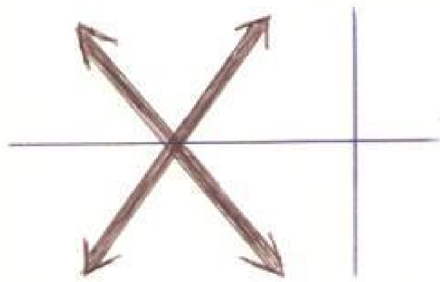
1 Asymptot



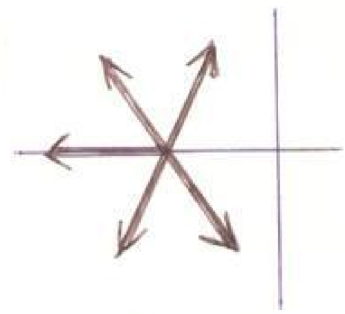
2 Asymptot



3 Asymptot



4 Asymptot



5 Asymptot

6 - The number of Asymptot in the root Locus equal to $(P-Z)$

7 - Those asymptotics intersect the real axis in one point (A) where

$$A = \frac{\sum \text{poles} - \sum \text{zeros}}{P - Z}$$

8 - Find the breakaway and break-in points. Because of the Conjugate symmetry of the root Loci, the breakaway points and break-in points either Lie on the real axis or occur in Complex-conjugate pairs.

If a root Locus Lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros.

Suppose that the c/c equation is given by

$$A(s) + K B(s) = 0$$

The breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{A'(s)\,B(s) - A(s)\,B'(s)}{B^2(s)}$$

If the value of K corresponding to root $s = s_1$ of $\frac{dK}{ds} = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. If the value of K is negative, then point $s = s_1$ is not a breakaway or break-in point.

9- Find critical value of K and the points where the root loci cross the imaginary axis using Routh critearion or by letting $s = j\omega$ in the c/c equation, equating both the real part and the imaginary part to zero, and solving for ω and K.

Ex: Find the critical value of K and points of ~~intersect~~ intersection between root Loci and imaginary axis for c/c equation below :-

$$s^3 + 3s^2 + 2s + K = 0$$

1st method:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & 0 \\ s^0 & K & 0 \end{array}$$

$\therefore$ critical $K = 6$

Solving the auxiliary equation obtained from the s^2 row.

$$3s^2 + K = 3s^2 + 6 = 0 \Rightarrow s^2 = -2 \Rightarrow s = \mp j\sqrt{2}$$

2nd method:

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

$$K - 3\omega^2 = 0$$

$$2\omega - \omega^3 = 0$$

for which $\omega = \mp\sqrt{2}$, $K = 6$

(10) - The angles of departure (or angles of arrival) of root Loci from the complex poles (or at complex zeros) is given by :

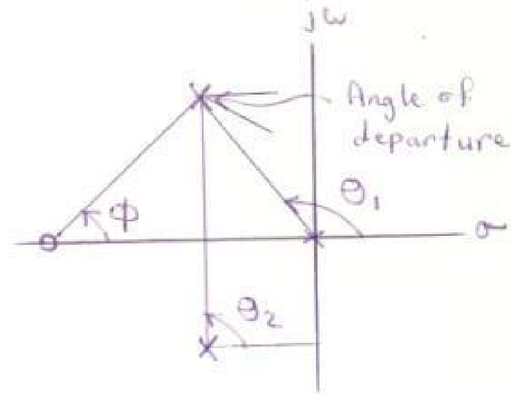
$$\theta_d = 180^\circ - \theta_p + \phi_z$$

Where

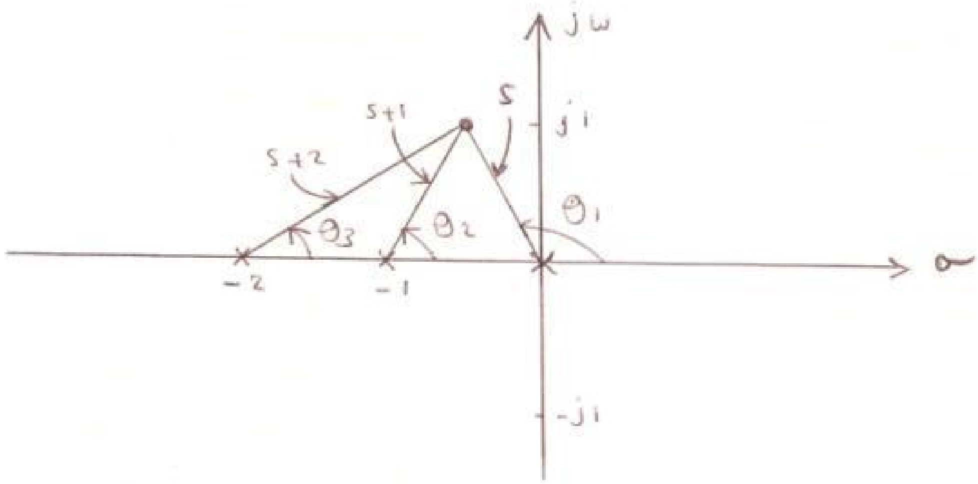
θ_d = angle of departure

θ_p = is the sum of the all angles subtended by other poles.

Φ_z = is the sum of the all angles subtended by Other Zeros.



(11). Choose a test point in the broad neighborhood of the jω axis and the origin, as shown in fig. below, and apply the angle Condition. If the test point is on the root Loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must by 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition.



Ex- $G(s) = \frac{K}{s(s+1)(s+2)}$, $H(s) = 1$, (K is non negative)

sketch the root-Locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant Complex-Conjugate closed-Loop poles is 0.5.

Solution:-

①. No. of Loci = 3

②. No. of asymptotes = 3

$$\text{Angles of asymptotes} = \frac{\mp 180(2n+1)}{P-Z} = \frac{\mp 180(2n+1)}{3}$$

$$= \mp 60(2n+1) \quad ; \quad (n = 0, 1, 2, \dots)$$

$$= 60^\circ, -60^\circ, 180^\circ$$

$$A = \frac{\sum \text{poles} - \sum \text{zeros}}{P-Z} = \frac{0+(-1)+(-2)}{3} = \frac{-3}{3} = -1$$

③- The c/c equation is

$$G(s) + 1 = 0 \implies \frac{K}{s(s+1)(s+2)} + 1 = 0$$

$$K = -(s^3 + 3s^2 + 2s)$$

$$\frac{dK}{ds} = 0 \implies \frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

$$s = -0.4226 \quad ; \quad s = -1.5774$$

$s = -0.4226$ is corresponds to the actual breakaway point.

$K = 0.3849$ for $s = -0.4226$

$K = -0.3849$ for $s = -1.5774$

④. The points where the root loci cross the imaginary axis are:

$$K = 6 \quad ; \quad s = \mp j\sqrt{2}$$

⑤. $\zeta = 0.5$; $\mp \cos^{-1}\zeta = \mp \cos^{-1} 0.5 = \mp 60^\circ$

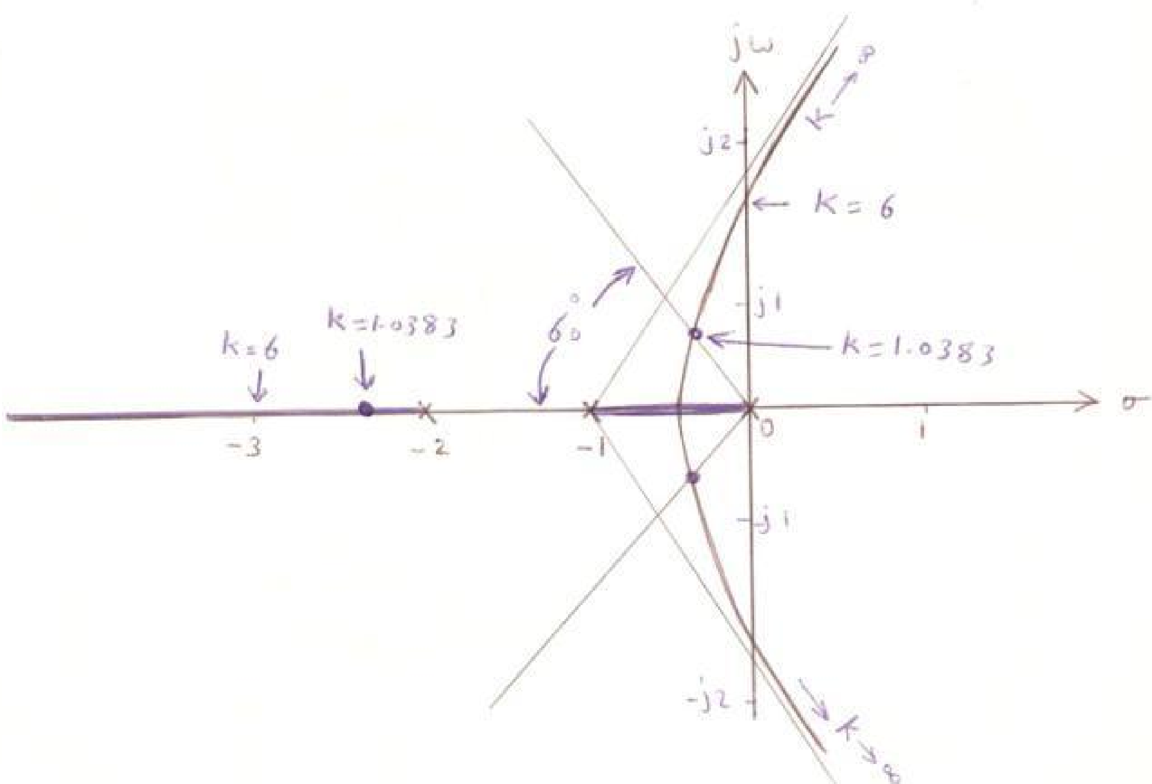
$s_1 = -0.3337 + j\,0.578$; $s_2 = -0.3337 - j0.578$

From magnitude condition

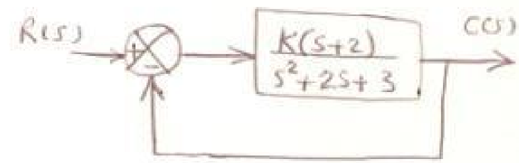
$$K = \left| s(s+1)(s+2) \right|_{s=-0.3337+j\,0.578}$$

$$K = 1.0383$$

Using this value of K, the third pole is found at $s = -2.3326$



Ex2: For the system shown in figure, sketch the root-locus plot.

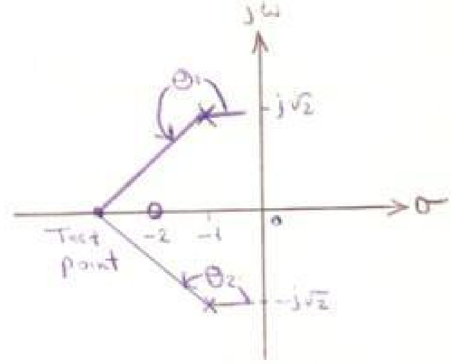


Solution: $G(s) = \frac{K(s+2)}{s^2+2s+3}$; $H(s) = 1$ where $K \geq 0$.

G(s) has a pair of complex conjugate poles at

$s = -1 + j\sqrt{2}$; $s = -1 - j\sqrt{2}$

The angle of departure from the complex-conjugate open-loop poles.



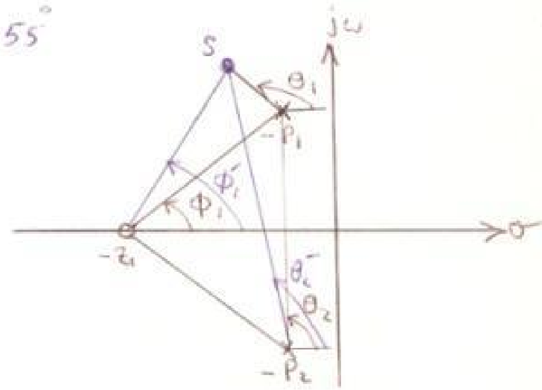
$$\phi_1' - (\theta_1 + \theta_2') = \mp 180(2n+1)$$

or

$$\theta_1 = 180 - \theta_2' + \phi_1' = 180 - \theta_2 + \phi_1$$

$$\theta_1 = 180 - \theta_2 + \phi_1 = 180^\circ - 90^\circ + 55^\circ$$

$$\theta_1 = 145^\circ$$



Determine the break-in point

$$K = -\frac{s^2+2s+3}{s+2}$$

$$\frac{dk}{ds} = -\frac{(2s+2)(s+2) - (s^2+2s+3)}{(s+2)^2} = 0$$

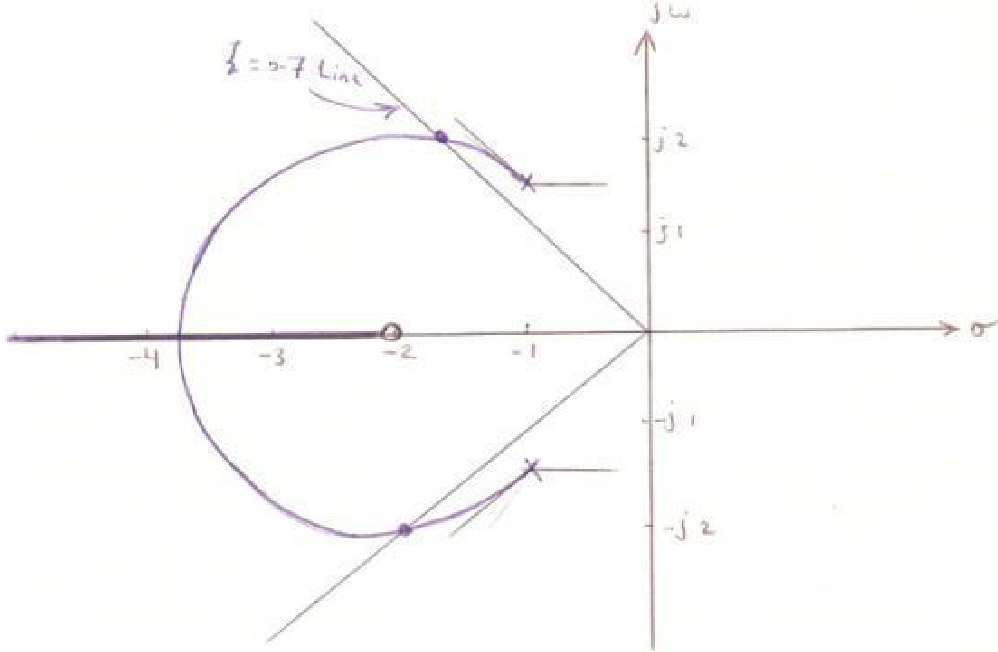
which gives $s^2 + 4s + 1 = 0$

or $s_1 = -3.7320$ or $s_2 = -0.2680$

hence $s = -3.7320$ is break-in point

and $K = 5.4641$ at this point.

(Note that at $s = -0.2680$; $K = -1.4641$)



The value of the gain K at any point on the root Locus can be found by applying the magnitude condition.

at $\zeta = 0.7$

$$K = \left| \frac{(s+1-j\sqrt{2})(s+1+j\sqrt{2})}{s+2} \right|_{s=-1.67+j1.7}$$

$$= 1.34$$

To show the occurrence of a circular root-Locus, we need to derive the equation for the root locus.

For this system, the angle condition is

$$\angle s+2 - \angle s+1-j\sqrt{2} - \angle s+1+j\sqrt{2} = \mp 180(2n+1)$$

if $s = \sigma + j\omega$, we obtain

$$\angle \sigma+2+j\omega - \angle \sigma+1+j\omega-j\sqrt{2} - \angle \sigma+1+j\omega+j\sqrt{2} = \mp 180(2n+1)$$

Which can be written as

$$\tan^{-1}\left(\frac{\omega}{\sigma+2}\right) - \tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right) - \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = \mp 180(2n+1)$$

or

$$\tan^{-1}\left(\frac{\omega-\sqrt{2}}{\sigma+1}\right) + \tan^{-1}\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma+2}\right) \mp 180^{\circ}(2n+1)$$

Taking tangents of both sides of this Last equation using the relationship $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

We obtain

$$\frac{\frac{\omega-\sqrt{2}}{\sigma+1} + \frac{\omega+\sqrt{2}}{\sigma+1}}{1 - \left(\frac{\omega-\sqrt{2}}{\sigma+1}\right)\left(\frac{\omega+\sqrt{2}}{\sigma+1}\right)} = \frac{\frac{\omega}{\sigma+2} \pm 0}{1 \mp \left(\frac{\omega}{\sigma+2}\right)(0)}$$

Which can be simplified as $\frac{2\omega(\sigma+1)}{(\sigma+1)^2 - (\omega^2-2)} = \frac{\omega}{\sigma+2}$

or $\omega\left[(\sigma+2)^2 + \omega^2 - 3\right] = 0$

This Last equation is equivalent to

$$\omega = 0 \quad \text{or} \quad (\sigma+2)^2 + \omega^2 = (\sqrt{3})^2$$

first equation $\omega = 0$, the real axis from $s = -2$ to $s = -\infty$

Second equation $(\sigma+2)^2 + \omega^2 = (\sqrt{3})^2$, the root Locus is a circular has radius equal $\sqrt{3}$ and Center at $\sigma = -2$ and $\omega = 0$, Where $K \geq 0$ (positive value).