❖ Differential Equation(D.E.)

The differential equation is a relation between two variables. It consists of the two or more variables and the derivatives of one variable to others.

In general, there are two types of differential equations which are:

1- Ordinary differential equation

This type has two variables which can be expressed as the following general equation:

$ax + by + c\,\bar{y} + d\bar{\bar{y}} + e\bar{\bar{\bar{y}}} + \dots\dots\dots + f\,y^{n}$ ………. (1)

Where a, b, c, ………. Are constants, and n is the order of derivative.

2- Partial differential equation

In this type, there are more independent variables and one dependent variable as illustrated in equation two.

$(\frac{dz}{dx}) + (\frac{d^2z}{dxdy}) +$ ………. (2)

There are some definitions that must be considered:

- ❖ Order of the differential equation, which represents the order of highest derivative.
- ❖ Degree of the differential equation, which is the highest exponent of the highest order derivative.

Ex_1/ what is the order and degree of the following differential equations:

1- $\bar{\bar{y}} + 4y = 3$ ………. (3)

2- $y + (\bar{\bar{\bar{\bar{y}}}})^3 + (\bar{y})^5=0$ ………. (4)

Sol:

The first equation has order = 2 and degree = 1& for the second equation order = 4 and degree = 3.

❖ First order D.E.

The general form of this type is

$F(x, y, \bar{y}) = 0$ .... (5)

There are different ways can be used to solve this type of equations, which are:

1- Separation of variables

$f(x)dx + f(y)dy = 0$ .... (6)

The solution is obtained in the form of integration as shown in the following equation:

$\int f(x)dx + \int f(y)dy = c$ .... (7)

Ex_2/ solve for y

1-) $2y\frac{dy}{dx} + 3 = 0$

2-) $(1 + x^2)\, dy - (xy)dx = 0$

3-) $\frac{dy}{dx} = \sqrt{xy}$

Sol:

1-) $2y\frac{dy}{dx} + 3 = 0 \rightarrow 2ydy + 3dx = 0$

$(2y^2)/2 + 3x = c \rightarrow y = \sqrt{c - 3x}$ where (c) is constant

2-) $(1 + x^2)\, dy - (xy)dx = 0$

$(1 + x^2)\, dy = (xy)dx \rightarrow \int(\mathrm{dy/y}) = \int(\mathrm{xdx}/(1 + \mathrm{x}^2))$

$ln(y) = (1/2)ln(1 + x^2) + c$ [$let\ c = ln(g)$]

$\ln(y) = ln\sqrt{(1 + \mathrm{x}^2)} + \ln(g) \longrightarrow y = g\sqrt{(1 + \mathrm{x}^2)}$ where (g) is constant

3-) $dy/dx = \sqrt{xy} \longrightarrow \frac{dy}{\sqrt{y}} = \sqrt{x}\, dx \longrightarrow y^{\frac{-1}{2}}dy = x^{\frac{1}{2}}dx \longrightarrow 2y^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}} + c$

$$y = \frac{1}{2}\sqrt{\frac{2}{3}x^{\frac{3}{2}} + c}$$

2- Homogeneous D.E.

The differential equation is called homogeneous and can be solved in this way if it satisfies the following condition:

$f(\gamma x,\gamma y) = f(x,y)$ .... (7)

This type is solved as follow:

$V = \frac{y}{x} \rightarrow y = Vx \rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

Ex3/ solve the following

$xdy = (y - x)dx$

Sol: $\frac{dy}{dx} = \frac{y\gamma - \gamma x}{x\gamma} = \frac{y-x}{x}$ then this equation is homogeneous

Let $V = \frac{y}{x} \rightarrow y = Vx \rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$

$\rightarrow V + x\frac{dV}{dx} = \frac{Vx-x}{x} \rightarrow V + x\frac{dV}{dx} = V - 1$

$x\frac{dV}{dx} = -1 \rightarrow dv = \frac{-dx}{x}$

$\int dv = \int \frac{-dx}{x} \rightarrow V = -\ln(x)$ +c but $V = \frac{y}{x}$ then

$\frac{y}{x} = -\ln(x) + c \rightarrow$ y $= -x\ln(x) + cx$

3- Equation Reducible to Homogeneous

If the condition of homogeneous equation is not satisfied, the differential equation can be reduced to homogeneous case as follow:

For the equation $\frac{dy}{dx} = \frac{(a_1x+b_1y+c_1)}{(a_2x+b_2y+c_2)}$ ... ... (8)

There are three cases:

- If c_1&c_2 =0 then this equation is homogeneous
- If c_1&$c_2 \neq 0$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ then let $z = a_1x + b_1y$
- If c_1& $c_2 \neq 0$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ then let $x = X + h$ & $y = Y + k$

This lead to

$$\frac{dy}{dx}=\frac{a_1X+b_1Y+a_1h+b_1k+c_1}{a_2X+b_2Y+a_2h+b_2k+c_2} \quad \text{……….} (9)$$

Putting $a_1h\ +b_1k+c_1=0$ & $a_2h\ +b_2k+c_2=0$ then find $(h\&k)$ and solve the D.E.

Ex₄/ for $\frac{dy}{dx}=\frac{2x+y+3}{4x+2y-5} \rightarrow \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$

Let $z\ =\ 2x+y$

$\frac{dz}{dx}=2+\frac{dy}{dx} \rightarrow \frac{dz}{dx}=\ 2\ +\frac{z+3}{2z-5}$

$\frac{dz}{dx}=\frac{4z-10+z+3}{2z-5} \rightarrow \frac{dz}{dx}=\frac{5z-7}{2z-5}$

$\frac{2z-5}{5z-7}\ dz=dx$ (using long division) $\rightarrow \int\frac{2}{5}dz-\int\frac{9}{5(5z-7)}dz=\int dx$

$\frac{2}{5}z-\frac{9}{25}\ln(5z-7)=x+c$ but $z\ =2x+y$ then

$\frac{2}{5}(2x+y)-\frac{9}{25}\ln(5(2x+y)-7)=x+c$

Ex₅/ Solve the following $\frac{dy}{dx}=\frac{x-y-2}{2x+y-1}$

Sol:

$\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1+2=3$

Let $x\ =\ X+h, y\ =Y+k \rightarrow \frac{dy}{dx}=\frac{X+h-Y-k-2}{2X+2h+Y+k-1}$

$$\begin{array}{r} h-k-2 \\ 2h+k-1 \\ \hline 3h\ -3 \end{array} \quad \text{Adding}$$

$\rightarrow h\ =1\&\ k\ =\ -1$

$\frac{dy}{dx}=\frac{X-Y}{2X+Y}$ this equation is homogeneous then let $V=\frac{y}{x} \rightarrow y\ =Vx$

$\frac{dy}{dx}=V+x\frac{dV}{dx} \rightarrow V+x\frac{dV}{dx}=\frac{x-Vx}{2x+Vx}$

$x\frac{dV}{dx}=\frac{1-V}{2+V}-V \rightarrow x\frac{dV}{dx}=\frac{1-V-2V+V^2}{2+V}$

$\int\frac{dx}{x}=\int\frac{(2+V)}{(1-3V-V^2)}dV$

$\int\frac{-dx}{x}=\int\frac{(2+V)}{(V^2+3V-1)}dV$

$\int\frac{-dx}{x}=\int\frac{(2+V)}{(V^2+3V-1+\frac{9}{4}-\frac{9}{4})}\ dV$

$\int\frac{-dx}{x}=\int\frac{(2+V)}{(V^2+3V+\frac{9}{4}-\frac{7}{2})}\ dV$

$\int\frac{-dx}{x}=\int\frac{(2+V)}{(V+\frac{3}{2})^2-\frac{7}{2})}\ dV$

Let $u=V+\frac{3}{2}\rightarrow V=u-\frac{3}{2}$ & $du\ =\ dV$ then

$\int\frac{-dx}{x}=\int u\mathrm{du}/(\mathrm{u}^2-\frac{7}{2})+\int\frac{1}{2}\mathrm{du}/(\mathrm{u}^2-\frac{7}{2})$

$\int\frac{-dx}{x}=\int u\mathrm{du}/(\mathrm{u}^2-\frac{7}{2})+\int\frac{1}{2}\mathrm{du}/(\mathrm{u}^2-\frac{7}{2})$

$-lnx=\frac{1}{2}\ln(\mathrm{u}^2-\frac{7}{2})-\frac{1}{2}tanh^{-1}(\sqrt{\frac{2}{7}}u)$

$\rightarrow -lnx=\frac{1}{2}\ln((V+\frac{3}{2})^2-\frac{7}{2})-\frac{5}{8}tanh^{-1}(\sqrt{\frac{2}{7}}(V+\frac{3}{2}))$

$ln\frac{1}{x}=\frac{1}{2}tanh^{-1}(\sqrt{\frac{2}{7}}(\frac{y}{x}+\frac{3}{2}))-\frac{1}{2}\ln((\frac{y}{x}+\frac{3}{2})^2-\frac{7}{2})$

4- Exact Equation

The differential equation that can be solved by this way must be satisfied the following condition:

For the D.E. $M(x,y)dx+N(x,y)dy=0$ the exact solution can be applied if $\frac{dM}{dy}=\frac{dN}{dx}$ and the solution of differential equation is

$F=\ \int Mdx+\int ndy$ ………. (10)

Or $F=\ \int Ndy+\int mdx$ ………. (11)

Where (n) represents all polynomials of (N) that don't have the variable (x), and (m) represents all polynomials of (M) that don't have the variable (y).

Ex$_6$/ Find the solution of the equation:

$$3x(xy - 2)\,dx + (x^3 + 2y)\,dy = 0$$

Sol:

$$M(x,y) = 3x(xy-2) \rightarrow \frac{dM}{dy} = 3x^2$$

$N(x,y) = x^3 + 2y \rightarrow \frac{dN}{dx} = 3x^2$ this mean that the D.E. is exact and can solved as follow:

$$F(x,y) = \int(3x^2y - 6x)dx + \int(2y)dy$$

$$F(x,y) = x^3y - 3x^2 + y^2 + c$$

HW_1: Obtain the same result using equation (11).

5- Linear Differential Equation

A First order linear differential equation is an differential equation of the form:

$$\bar{y} + P(x)y = Q(x)\ linear\ in\ (y)$$

$$\bar{x} + \propto(y)x = \beta(y)\ \ linear\ in\ (x)\ \ \ldots\ldots\ldots\ (12)$$

Where P and Q are functions of (x). This form is called the standard form. The equation is called first order because it only involves the function (y) and first derivatives of (y). Such equations can be solved as follow:

$y = \frac{1}{\rho}\int \rho\, Q(x)dx$ ………. (13) Where $\rho = e^{\int P(x)dx}$

Ex$_7$/ Solve the following D.E.

$$\frac{dy}{dx} + 3ycosx = cosx$$

Sol:

$\rho = e^{\int 3cosxdx} = e^{3sinx}$

$$y = \frac{1}{e^{3sinx}}\int e^{3sinx}cosx\, dx \rightarrow y = \frac{1}{e^{3sinx}}\left[\frac{1}{3}e^{3sinx} + c\right] = \frac{1}{3} + ce^{-3sinx}$$

❖ Bernoulli's Equation

The general form of this equation is:

$\frac{dy}{dx} + u(x)y = v(x)\, y^n$ ………. (14)

To solve this equation, let $z = y^{1-n} \rightarrow \frac{dz}{dx} = (1-n)y^{-n}\,\frac{dy}{dx}$

Multiply eq. (14) by $(1-n)y^{-n} \rightarrow$

$$(1-n)y^{-n}\,\frac{dy}{dx} + (1-n)u(x)yy^{-n} = (1-n)v(x)y^{-n}y^n$$

$\frac{dz}{dx} + (1-n)u(x)y^{1-n} = (1-n)v(x)$ ………. (15)

Ex$_8$/ Solve the following differential equation $\frac{dy}{dx} = xy + xy^4$

Sol: From equation

$$\frac{dz}{dx} + (1-n)u(x)y^{1-n} = (1-n)v(x)$$

$$\frac{dz}{dx} + 3xy^{-3} = -3 \rightarrow \frac{dz}{dx} + 3xz = -3x$$

$$\rho = e^{\int 3x\,dx} = e^{\frac{3}{2}x^2}$$

$$z = \frac{1}{e^{\frac{3}{2}x^2}} \int e^{\frac{3}{2}x^2} * -3x dx$$

$$= \frac{-1}{e^{\frac{3}{2}x^2}} \left[e^{\frac{3}{2}x^2} + c \right] \rightarrow z = -\,c\, e^{\frac{-3}{2}x^2} - 1$$

$$\rightarrow y^{-3} = -c\, e^{\frac{-3}{2}x^2} - 1 \;\therefore\; y = \sqrt[3]{\frac{-1}{c\, e^{\frac{-3}{2}x^2} + 1}}$$

❖ **Special Case of First-Order Linear D.E.**

For the following $f(y)\,\frac{dy}{dx} + p(x) * g(y) = Q(x)$, the solution can be performed as:

$V = g(y) \rightarrow \frac{dv}{dx} = \frac{d[g(y)]}{dx} . \frac{dy}{dx}$ ………. (16)

Ex9/ Solve the following D.E. $\frac{1}{1+y^2}\,\frac{dy}{dx} + \frac{1}{x} tan^{-1}y = x$

Sol:

$$\frac{dy}{dx} + \frac{1}{x}(1 + y^2)tan^{-1}y = x(1 + y^2)$$

$$f(y) = \frac{1}{1+y^2} \;, g(y) = tan^{-1}y \;,\; Q(x) = x$$

$$V = tan^{-1}y \rightarrow \frac{dv}{dx} = \frac{1}{1+y^2}\,\frac{dy}{dx}$$

$$\rightarrow \frac{1}{1+y^2}(1 + y^2)\frac{dV}{dx} + \frac{1}{x}\, tan^{-1}y = x$$

$$\frac{dV}{dx} + \frac{V}{x} = x$$

$\rho = e^{\int \frac{1}{x} dx} \rightarrow \rho = x \rightarrow V = \frac{1}{x}\left[\int x^2 dx\right] \rightarrow V = \left[\frac{1}{3}x^2 + \frac{c}{x}\right]$

$\rightarrow tan^{-1}y = \frac{1}{3}x^2 + \frac{c}{x}$ or $y = \tan\left[\frac{1}{3}x^2 + \frac{c}{x}\right]$

❖ Second – Order D.E.

There are many forms of the second order differential equations which are:

1- Equation immediately integrable

The general form of this type is

$\frac{d^2y}{dx^2} = f(x)$ ………. (17)

Which can be solved by two integration, the first integration gives

$\frac{dy}{dx} = \int f(x)dx + c_1$ ………. (18)

A second integration gives a general solution for this case:

$y = \int(\int f(x)dx + c_1x + c_2$ ………. (19)

Ex_{10} / Solve the following D.E.

$d^2y = (3x - 2)dx^2$

Sol:

$$\frac{d^2y}{dx^2} = 3x - 2$$

$\frac{dy}{dx} = \frac{3}{2}x^2 - 2x + c_1 \rightarrow y = \frac{1}{2}x^3 - x^2 + c_1x + c_2$

2- Second – Order D.E. not containing (y)

The equation form of this type is:

$F(x, \bar{y}, \bar{\bar{y}}) = 0$ ………. (20)

Let $p = \frac{dy}{dx} \rightarrow \frac{dp}{dx} = \frac{d^2y}{dx^2}$

Ex$_{11}$/ Solve $(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

Sol:

Let $p=\frac{dy}{dx} \rightarrow \frac{dp}{dx}=\frac{d^2y}{dx^2}$

$\rightarrow (1+x)\frac{dp}{dx}+p=0$. Which exact and can be solved as

$(1+x)p=c_1$ but $p=\frac{dy}{dx}$ then $dy=\frac{c_1dx}{(1+x)}$

$\rightarrow y=c_1\ln(1+x)+c_2$

3- Second – Order D.E. not containing (x)

In this case the differential equation is

$\frac{d^2y}{dx^2}=f\left(y,\frac{dy}{dx}\right)$ ………. (21)

Substitute $p=\frac{dy}{dx}$ and write the second derivative in the form

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=p\,\frac{dp}{dy}$$

This lead $p\,\frac{dp}{dy}=f(y,p)$

Ex$_{12}$/ Solve the following differential equation

$y\,\frac{d^2y}{dx^2}=y^2\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2$

Sol:

Let $p=\frac{dy}{dx} \rightarrow \frac{d^2y}{dx^2}=p\,\frac{dp}{dy}$

$\rightarrow yp\frac{dp}{dy}=y^2p+p^2$ [linear differential equation]

The solution of this equation is:

$\frac{p}{y}=y+c_1$

$\rightarrow \frac{dy}{dx} = y(y + c_1) \rightarrow \frac{dy}{dx} - yc_1 = y^2$

Use Bernoulli's Solution

let $z = y^{(1-2)} \rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \rightarrow -y^{-2}\frac{dy}{dx} - yc_1(-y^2) = y^2 * (-y^{-2})$

$\frac{dz}{dx} + c_1 z = -1 \rightarrow \frac{dz}{dx} = -1 - c_1 z \rightarrow \frac{dz}{(1+c_1 z)} = dx$ integrate the two sides

$\frac{\ln(1+c_1 z)}{c_1} = -x + c_2 \rightarrow \ln(1 + c_1 z) = -c_1 x + c_1 c_2$(taking Exp.) $\rightarrow$

$1 + c_1 z = e^{-c_1 x + c_1 c_2}$, but $z = y^{-1} \rightarrow y^{-1} = \frac{e^{-c_1 x + c_1 c_2 - 1}}{c_1}$

$y = \frac{c_1}{e^{-c_1 x + c_1 c_2 - 1}}$

❖ <u>Second – Order Homogenous D.E.</u>

The general form of the second-order homogeneous D.E. is:

$a\bar{\bar{y}} + b\bar{y} + cy = 0$ ………. (22)

To solve this equation let $y = e^{rx} \rightarrow \bar{y} = re^{rx} \rightarrow \bar{\bar{y}} = r^2 e^{rx}$ this lead to:

$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$ ………. (23) since $e^{rx} \neq 0$ then

$ar^2 + br + c = 0$ ………. (24)

Now the roots of eq.(24) and according to these roots the solution of eq.(22) will be:

- If r_1 and r_2 are equal real numbers then $y = e^{rx}(c_1 + xc_2)$
- If r_1 and r_2 are not equal real numbers then $y = (c_1 e^{r_1 x} + c_2 e^{r_2 x})$
- If r_1 and r_2 are complex conjugated numbers then

$y = e^{\propto x}(c_1 cos\beta x + c_2\, sin\beta x)$

Where $(c_1, c_2)$are constants and $(\propto, \beta)$ are the real and imaginary parts of the complex root.

Ex13/ Solve $\bar{\bar{y}} - 6\bar{y} + 13y = 0$

Sol:

Let $y = e^{rx} \rightarrow \bar{y} = re^{rx} \rightarrow \bar{\bar{y}} = r^2 e^{rx}$

$r^2 e^{rx} - 6\, re^{rx} + 13e^{rx} = 0$ since $e^{rx} \neq 0$ then

$r^2 - 6\,r + 13 = 0$

$r = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$

$\rightarrow r = 3 \pm j2$

$y = e^{3x}(c_1 cos2x + c_2\, sin2x)$

❖ <u>Non-Homogeneous 2nd Order D.E.</u>

Consider a non homogeneous linear equation

$a_n\, y^n + a_{n-1} y^{n-1} + \ldots\ldots + a_0 y = g(x)$ ………. (25)

The general solution of such equation is of the form

$y = y_h + y_p$

where y_h is the general solution of homogeneous equation and y_p is called the particular solution and depends on the non homogeneous part.

To solve this equation the following steps is applied

1- Find y_h as illustrated previously
2- Find y_p as will be illustrated
3- The final solution is $y = y_h + y_p$

The methods that can be used to find the particular solution are:

❖ Undetermined coefficients

This method is useful when the differential equation has constant coefficients and the function $g(t)$ has a special form: some linear combination are [A, t^n; $e^{\propto t}$, $e^{\propto t}$cos(βt); $e^{\propto t}$sin(βt)]: where A is constant.

- Remark[1]: For equation (24) If $g(x)$ is constant then $y_p = A$

Ex14/ Solve the following differential equation

$\bar{\bar{y}} - 3\bar{y} - 10y = 3$

Sol:

$r^2 - 3r - 10 = 0 \;\; \rightarrow (r-5)(r+2) = 0$

$r_1 = 5, r_2 = -2 \rightarrow y_h = \; c_1\, e^{5x} + c_2\, e^{-2x}$

$y_p = A \;\; \rightarrow \bar{\bar{y}}_p = 0 \rightarrow \;\bar{y}_p = 0$

$\rightarrow 0 - 3 * 0 - 10A = 3 \rightarrow A = \frac{-3}{10} \rightarrow y_p = \frac{-3}{10}$

$\therefore\, y = \dfrac{-3}{10} + \, c_1\, e^{5x} + c_2\, e^{-2x}$

- Remark2: For equation (24) If g(x) is a function of (x)then

  $y_p = Q(x)$

Where $Q(x)$ is a polynomial $(a + bx + cx^2 + dx^3 \ldots\ldots + x^n)$,then $y_p = Ax^n + \;\ldots . + Bx^2 + Cx + D$

Ex15/ Find y_p for the following second order D.E.

$\bar{\bar{y}} + 2\bar{y} - \; 2y \;= 2x^2$

Sol:

Since $Q(x) = \; 2x^2 \rightarrow y_p = Ax^2 + \; Bx + C \;\; \rightarrow \bar{y}_p = (2Ax + B)$

& $\bar{\bar{y}}_p = 2A$

$\therefore\, 2A + 2(2Ax + B) - 2\,(Ax^2 + \; Bx + C\,) = \; 2x^2$

$\rightarrow 2A + 4Ax + 2B - 2\,Ax^2 - 2\,Bx - 2C = \; 2x^2 \;\; \rightarrow \; -2A = 2$

$\therefore\, A = -1$

, $4A - 2B = \; 0 \; \rightarrow B = -2$ & $2A + 2\,B - 2C = 0 \; \rightarrow C = \; -3$

This means that $y_p = \; -x^2 - 2x - 3$

- Remark3: For equation (24), If g(x) = $p(x)e^{\propto x}$ where $p(x)$ is a polynomial with degree (m) then $y_p = Q(x)e^{\propto x}$ if ($\propto$) is not root of homogeneous solution, where $Q(x)$ is a polynomial with degree

(m). and if $\propto$ is a simple root of homogeneous solution then $y_p = xQ(x)e^{\propto x}$, while if $\propto$ a root of multiplicity (n) of homogeneous solution then $y_p = x^n\, Q(x)e^{\propto x}$.

- Remark$_4$: If g(x) = $kcos\beta x$ or g(x) =$ksin\beta x$, if $j\beta$ is not a simple root of homogeneous solution then y_p =$Mcos\,\beta x + Nsin\beta x$ where $M\&N$ are constants, and if $j\beta$ is a simple root of homogeneous solution then y_p =$x(\ Mcos\,\beta x + Nsin\beta x)$, while if $j\beta$ a root of multiplicity (n) of homogeneous solution then $y_p = x^n(\ Mcos\,\beta x + Nsin\beta x)$.
- Remark$_5$: If g(x) = $e^{\propto x}[P(x)cos\beta x + Q(x)sin\beta]$ and $(\propto +j\beta)$ is not a root of homogeneous solution then y_p =$e^{\propto x}(\ U(x)cos\,\beta x + V(x)sin\beta x)$. Where $U(x)\&V(x)$ are polynomials of a degree equal to the highest degree of $(P(x)\&\,Q(x))$.

  While if g(x) = $e^{\propto x}[P(x)cos\beta x + Q(x)sin\beta]$ and $(\propto +j\beta)$ is a root of multiplicity (n) homogeneous solution then y_p =$x^n e^{\propto x}(\ U(x)cos\,\beta x + V(x)sin\beta x)$. Where $U(x)\&V(x)$ are polynomials of a degree equal to the highest degree of $(P(x)\&\,Q(x))$.

Ex$_{16}$/ Solve the following D.E.

$\bar{\bar{y}} - \bar{y} = \cos(2x)$

Sol: $r^2 - \ r = 0 \ \rightarrow (r-1)\,r = 0 \ \rightarrow \ r_1 = 0\ or\ r_2 = 1$

$y_h = (c_1 + c_2 e^x)$

$y_p = Mcos\beta x + Nsin\beta x$

$\bar{y}_p = -M\beta sin\beta x + N\beta cos\beta x$

$\bar{\bar{y}}_p = -M\beta^2 cos\beta x - NB\beta^2 sin\beta x$

$-M\beta^2 cos\beta x - N\beta^2 sin\beta x + M\beta sin\beta x - N\beta cos\beta x = \ \cos(2x)$

Since (β=2) then

$-4Mcos(2x) - 4Nsin(2x) + 2Msin(2x) - 2Ncos(2x) = \cos(2x)$

$\rightarrow -4(M-N)\cos(2x) + 2(M-N)\sin(2x) = \cos(2x)$

$\therefore -4M - 2N = 1 \ldots..(i)$

$\underline{2M - 4N \ = 0 \ldots} \ldots (ii)$ Multiply equation (ii) by (2) then adding eq. (i) &

$-10N = 1$

$\rightarrow N = -\frac{1}{10} \rightarrow M = \frac{-1}{5} \quad \therefore y_g = (c_1 + c_2 e^x) - \frac{1}{10} cosBx - \frac{1}{5} sinBx$

- **Determined coefficients**

The general form of the second order D.E. with constant coefficient is

$a\bar{\bar{y}} + b\,\bar{y} + c\,y = g(x)$ .... (26)

To solve eq. (25) the following equations must be considered.

$y_p = U_1 V_1 + U_2 V_2$ & $y_h = C_1 V_1 + C_2 V_2$, to find the solution of eq. (25) solve the following

$\bar{U}_1 V_1 + \bar{U}_2 V_2 = 0$ & $\bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 = g(x)$. Then find $\bar{U}_1$ & $\bar{U}_2$

As follow:

$$\bar{U}_1 = \frac{\begin{vmatrix} 0 & V_2 \\ g(x) & \bar{V}_2 \end{vmatrix}}{\begin{vmatrix} V_1 & V_2 \\ \bar{V}_1 & \bar{V}_2 \end{vmatrix}} , \bar{U}_2 = \frac{\begin{vmatrix} V_1 & 0 \\ \bar{V}_1 & g(x) \end{vmatrix}}{\begin{vmatrix} V_1 & V_2 \\ \bar{V}_1 & \bar{V}_2 \end{vmatrix}}$$

In addition, find U_1 & U_2 by integrating $\bar{U}_1$ & $\bar{U}_2$

Ex17/ Solve $\bar{\bar{y}} + 2\bar{y} - 3y = e^{2x}$

Sol:

$r^2 + 2r - 3 = 0 \rightarrow (r+3)(r-1) = 0$

$\therefore r_1 = -3$ & $r_2 = 1$

$\rightarrow y_h = c_1 e^{-3x} + c_2 e^x$

$y_p = U_1 V_1 + U_2 V_2$

$\bar{U}_1 V_1 + \bar{U}_2 V_2 = 0$& $\bar{U}_1 \bar{V}_1 + \bar{U}_2 \bar{V}_2 = g(x)$

In this problem ($V_1 = e^{-3x} \rightarrow \bar{V}_1 = -3e^{-3x}, V_2 = e^x \rightarrow \bar{V}_2 = e^x$)

$$\overline{U}_1 \; e^{-3x} + \overline{U}_2 \, e^{x} = 0$$

$$\underline{\pm 3\overline{U}_1 \;\; e^{-3x} \mp \overline{U}_2 \, e^{x} = \mp e^{2x}}$$

$$4\overline{U}_1 \, e^{-3x} \; = \; -e^{2x}$$

$\rightarrow \overline{U}_1 \;=\; \frac{-1}{4} e^{5x} \rightarrow U_1 \;=\; \frac{-1}{20} e^{5x}$ this lead to $U_2 \;=\; \frac{1}{4} e^{x}$

$\therefore \; y_g \;=\; c_1 \, e^{-3x} + c_2 \, e^{x} + \frac{1}{20} \, e^{2x} + \frac{1}{4} \, e^{2x}$

❖ <u>Higher – Order D.E.</u>

The higher order D.E. is given by equation (27)

$a_n \, \frac{d^n y}{dx^n} + \; a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \; \dots \dots .. + a_0 y = g(x)$ ………. (27)

To find y_h let eq. (26) equal to zero and to find y_p then

$y_p \;=\; U_1 \, V_1 + U_2 \, V_2 \;+\; \dots \dots \dots . + U_n \, V_n$ then

$$\overline{U}_1 V_1 + \overline{U}_2 \, V_2 \;+\; \overline{U}_2 \, V_3 + \; \dots \dots \dots \dots . \, \overline{U}_n \, V_n \;\; = 0$$

$$\overline{U}_1 \, \overline{V}_1 + \overline{U}_2 \, \overline{V}_2 + \; \overline{U}_3 \, \overline{V}_n + \; \dots \dots \dots \dots \; \overline{U}_n \, \overline{V}_n = 0$$

$$\vdots$$

$\overline{U}_1 \, V_1{}^{(n-1)} + \overline{U}_2 V_2{}^{(n-1)} \;+\; \overline{U}_2 \, V_3{}^{(n-1)} \;+\; \dots \dots \dots \dots \; \overline{U}_n \, V_n{}^{(n-1)} = g(x)$

Ex18/ Find the solution of $\frac{d^3 y}{dx^3} - 3\overline{y} + 3 = e^{x}$

Sol: $(r^3 - 3r + 2) = 0$

$\therefore \; r_1 = 1, r_2 = -2$ and $r_3 = 1$

$\rightarrow y_h \;=\; c_1 e^{x} + c_2 e^{-2x} + c_3 \, x e^{x}$

$y_p \;=\; U_1 \, V_1 + U_2 \, V_2 \; + U_3 \, V_3$

$$\overline{U}_1 \;\; e^{x} + \overline{U}_2 \, e^{-2x} + \; \overline{U}_3 \;\; x e^{x} = 0$$

$$\overline{U}_1 \;\; e^{x} - 2\overline{U}_2 \, e^{-2x} + \; \overline{U}_3 \;\; (x e^{x} + e^{x}) \;\; = 0$$

$$\overline{U}_1 \;\; e^{x} + 4\overline{U}_2 \, e^{-2x} + \; \overline{U}_3 \;\; (x e^{x} + e^{x} + e^{x}) = e^{x}$$

$$\begin{bmatrix}\bar{U}_1\\ \bar{U}_2\\ \bar{U}_3\end{bmatrix}\begin{bmatrix}e^{x} & e^{-2x} & xe^{x}\\ e^{x} & -2e^{-2x} & xe^{x}+e^{x}\\ e^{x} & 4e^{-2x} & xe^{x}+2e^{x}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ e^{x}\end{bmatrix}$$

HW$_2$: From above matrices, $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$ can be found, then integrated them to find U_1, U_2, and U_3 ,finally y_p & y_g can be found.

Applications of Differential Equations

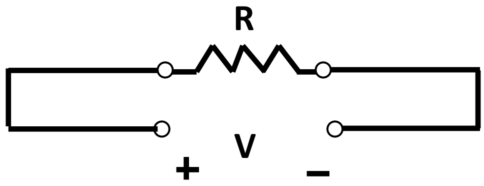
In the following section some of D.E. applications are considered, Some of these applications are:

1- Modeling Electrical Circuits

This application of the first-order differential equations arises in the electrical circuits.

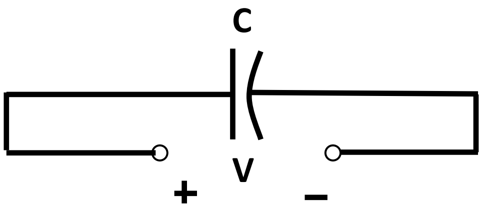
First, the current equations of the resistance, capacitance, and inductor must be considered.

1- Resistor



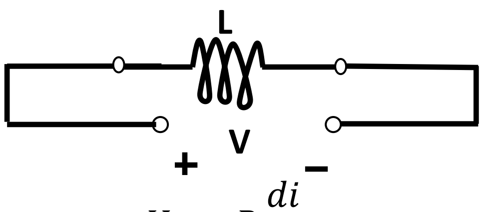
$V_R = IR$

2- Capacitor



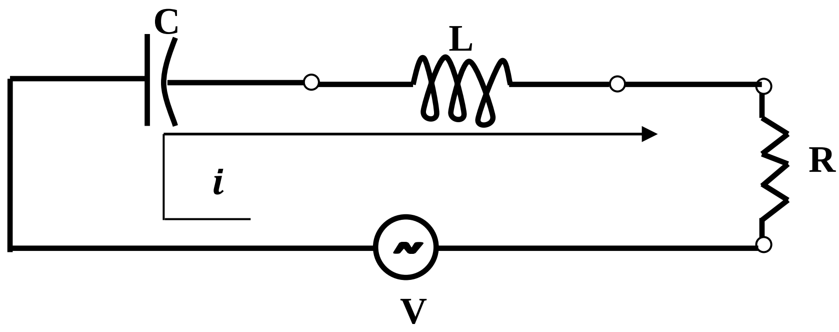
$$V_R = \frac{1}{c}\int i\ dt$$

3- Inductor



$$V_R = R\frac{di}{dt}$$

Ex19/ For the electrical circuit shown in the following figure find (i) if $V = 2\sin(t)$, $C = 46F$, $L = 0.01H$, and $R = 4k\Omega$.



Sol:

Appling K.V.L. then:

$$V = \frac{1}{C}\int i\,dt + L\,\frac{di}{dt} + Ri$$

$$\rightarrow 2\sin t = \frac{1}{4*10^{-6}}\int i\,dt + 0.01\frac{di}{dt} + 4i$$

Derive the two sides with respect to (t)

$$2cost = \frac{1}{4*10^{-6}} i + 0.01\frac{d^2i}{dt^2} + 4000\frac{di}{dt}$$

For simplicity, use the symbols of $(C, L, and\ R)$ then

$$\mathrm{L}\,\frac{d^2i}{dt^2} + R\,\frac{di}{dt} + \frac{1}{C}\,i = 2cost$$

$$\bar{\bar{\imath}} + \frac{R}{L}\,\bar{\imath} + \frac{1}{LC}\,i = 2cost$$

$$r^2 + 400000r + 25000000 = 0$$

$$\rightarrow r_1 = -399937.5\ \&\ r_2 = -62.51$$

$$i_h = C_1\,e^{-399937.5x} + C_2\,e^{-62.51x}$$

Since $g(x) = 2\sin(t)$ then

$$i_p = M\,cost + Nsint \ \rightarrow \ \bar{\imath}_p = -M\,sint + Ncost$$

$$\bar{\bar{\imath}}_p = -M\,cost - Nsint$$

$$\rightarrow -M\,cost - Nsint + 4*10^5(-M\,sint + Ncost) + 25*10^6\,(M\,cost + Nsint\ ) = 2sint$$

$-N - 4 * 10^5 M + 25 * 10^6 N = 2 \ldots\ldots\ldots (i)$

$-M + 4 * 10^5 N + 25 * 10^6 M = 0 \ldots\ldots\ldots (ii)$

$24999999N - 4 * 10^5 M = 2 \ldots\ldots\ldots (i)$

$4 * 10^5 N + 24999999M = 0 \ldots\ldots\ldots (ii)$

Solving equations (i) & (ii) to gate

$N \cong 8 * 10^{-8}$

$M \cong -0.13 * 10^{-8} \rightarrow i_p = -0.13 * 10^{-8}\cos(t) + 8 * 10^{-8}\sin(t)$

$\therefore i = C_1\, e^{-399937.5x} + C_2\, e^{-62.5} - 0.13 * 10^{-8} cost + 8 * 10^{-8}\sin(t)$

Another form of differential equation for the electrical circuit using the charge instead of current

Since for series RCL circuit the differential equation is

$RI + L\frac{di}{dt} + + \frac{1}{C}q = V(t) \ldots\ldots\ldots (28)$

But $I = \frac{dq}{dt} \rightarrow \frac{di}{dt} = \frac{d^2q}{dt^2}$, substituting in eq.(28) and dividing by (L)

$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{CL}q = \frac{V(t)}{L} \ldots\ldots\ldots (29)$

The initial conditions are: $q(0) = q_o$ and $\left.\frac{dq}{dt}\right|_{t=0} = I(0) = I_o$

And for current, the same D.E. of Ex19 is used by dividing it on (L)

$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}I = \frac{dV(t)}{Ldt}$

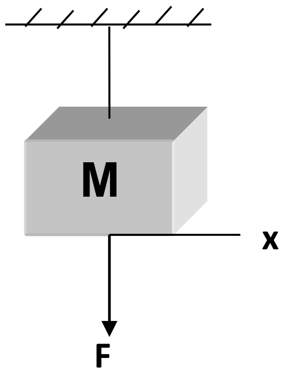
The initial conditions are: $I(0) = I_o$ and $\left.\frac{dI}{dt}\right|_{t=0} = \frac{V(0)}{L} - \frac{R}{L}I_o - \frac{1}{CL}q_o$

2- Modeling Free Mechanical Oscillations

The simple mechanic elements are:

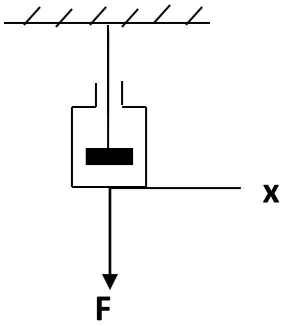
1- Mass

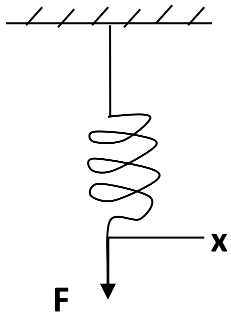
$F = M\,\bar{\bar{x}}$



2- Damping

$F = b\,\bar{x}$

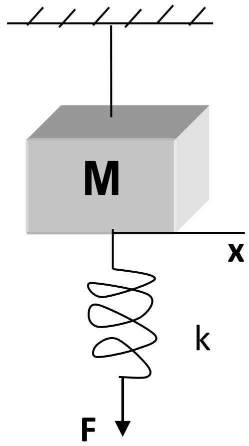




3- Spring

$F = kx$

Ex20/ For the mechanical system shown in the following figure, write the differential equation and solve it if ($F = 2e^t\ N$)



Sol:

$F = M\bar{\bar{x}} + kx$

$\rightarrow M\bar{\bar{x}} + kx = 2e^t$

$\bar{\bar{x}} + \frac{k}{M}x = \frac{2}{M}e^t$

$r^2 + r = 0$

$r(r+1) = 0 \ \rightarrow r_1 = 0\ \&\ r_2 = -1$

$x_h = C_1 + C_2\, e^{-t}$

$x_p = Ae^t \rightarrow \bar{x}_p = Ae^t \rightarrow \bar{\bar{x}}_p = Ae^t$

$\rightarrow\ Ae^t + \frac{k}{M}\,Ae^t = \frac{2}{M}e^t$

$\therefore A\left(1 + \frac{k}{M}\right) = \frac{2}{M} \ \rightarrow A = \frac{2}{M+k}$

$$\rightarrow\ x_p = \frac{2}{M+k}e^t$$

$$\therefore x = C_1 + C_2\, e^{-t} + \frac{2}{M+k}e^t$$

3- Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A.

Ex21/ When *60* grams of A are present, and after one hour only *10* grams of A remain unconverted into substance B. How much of A is present after two hours?

Sol:

Let y be the unconverted amount of substance A at any time t. the differential equation:

$$\frac{dy}{dt} = ky^2$$

$$\rightarrow \frac{dy}{y^2} = kdt \rightarrow \int \frac{dy}{y^2} = k\int dt$$

$$-\frac{1}{y} = kt + C$$

$$\rightarrow y = \frac{-1}{kt+C}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 60$ when $t = 0$ you can determine that $C = \frac{-1}{60}$ Similarly, since $y = 10$ when $t = 1$, it follows that $[10 = \frac{-1}{k*1-\frac{1}{60}}] \rightarrow k = \frac{-1}{12}$

$\rightarrow y = \frac{60}{5t+1}$, Now after two hours $y = \frac{60}{5*2+1} \rightarrow y = 5.45$ grams

There are another applications such as (Modeling Advertising Awareness, Modeling Population Growth, Modeling a Chemical Mixture, ……., etc.)

❖ Additional problems

1- $\bar{y} + \mathrm{y}\sin(x) = 0 \;\; for\; y\left(\frac{\pi}{2}\right) = 1$

2- $e^{x-y}\bar{y} + e^{y-x} = 0\; if\; y(0) = 0$

3- $\frac{y}{x} + \frac{dy}{dx}\ln(x) = 0$

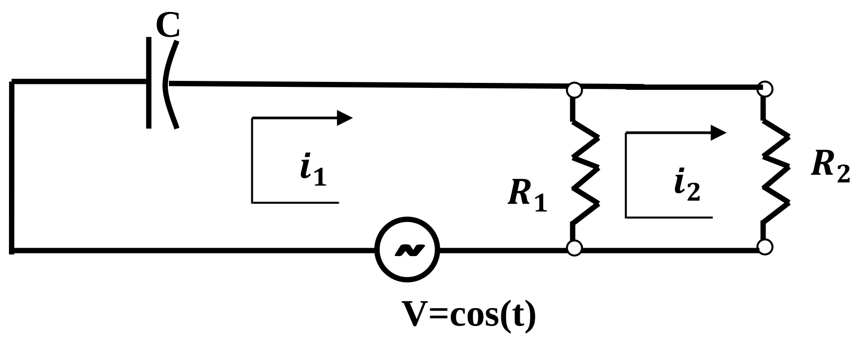
4- $[\cos(\theta) + 2r\sin^2(\theta)]dr + rsin(\theta)(2rcos(\theta) - 1)d\theta = 0$

5- $\bar{x} + \mathrm{x}\, sec^2(y^2) = \frac{y}{cos^2(y^2)}$

6- $\bar{y} + \mathrm{y}\sin(x) = y^3\sin(x) \;\; for\; y(0) = 2$

7- $\bar{y}\sin(y) + \mathrm{x}\tan(\mathrm{y})\sin(y) = \frac{\sec(y)}{e^{\mathrm{x}^2}}$

8- For the following electrical cct., find the current in each branch



9- Solve the following differential equation $\bar{\bar{y}} - \bar{y} = \sinh(2x)$