Solution of Differential Equations Using Power Series

❖ Power Series

These series are examples of infinite series where each term contains a variable (*x*) raised to a positive integer power. The most important statement one can make about a power series is that there exists a number (*R*) called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent.

The relation $|x| < R$ is equivalent to $-R < x < R$. At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent. To test convergence of Power Series consider the following statements

- The series converges absolutely if $|x| < R$
- The series diverges if $|x| > R$
- The series may be convergent or divergent at $x = \pm R$

Ex$_1$/ Find the radius of convergence for the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\ldots\ldots\ldots$$

Sol:

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots\ldots\ldots\ldots \sum_{n=0}^{\infty} \frac{x^n}{n+1}$$

So that $a_n = \frac{x^n}{1+n} \rightarrow a_{n+1} = \frac{x^{n+1}}{2+n}$

$$\therefore R = \lim_{n\to\infty} \frac{x^n x}{2+n} * \frac{1+n}{x^n}$$

$R = x$

If $|x| < 1$ *then the serise is conv. while if* $|x| > 1$ *then it is div.*

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There are two types of power series, which are:

1. **Maclaurin series ($M.S.$)**
2. **Taylor's series ($T.S.$)**

To find ($M.S.$) for $f(x), nth$ derivatives ($f^n(0)$) are performed then the rule that given in equation (1) is applied.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \quad \ldots \ldots (1)$$

Ex$_2$/ Find $M.S.$ for $[e^{3x}]$

Sol:

$f(x) = e^{3x} \rightarrow f(0) = 1$

$\bar{f}(x) = 3\, e^{3x} \rightarrow \bar{f}(0) = 3$

$\bar{\bar{f}}(x) = 9e^{3x} \rightarrow \bar{\bar{f}}(0) = 9$

$\bar{\bar{\bar{f}}}(x) = 27e^{3x} \rightarrow \bar{\bar{\bar{f}}}(0) = 27$

⋮

$f^n(x) = 3^n e^{3x} \rightarrow f^n(0) = 3^n$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n = 1 + 3x + \frac{9}{2} x^2 + \frac{27}{6} x^3 + \ldots \ldots$$

Hw$_1$: Find the M.S. for the following functions

1-) $\sinh x$

2-) $\cos x$

3-) $\ln(x)$

To find the Taylor series for any function, the equation (2) is applied:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n \quad \ldots \ldots (2)$$

❖ **Important Remarks**

- Remark$_1$: The function is said to be analytic at a point x_o if it has Taylor series at $x = x_o$ and it is said to be non – analytic if Taylor series does not exist at ($x = x_o$).

- Remark$_2$: For the homogeneous D.E. that given in equation (3),

$$y^n(x) + a_{n-1}(x)y^{n-1}(x) + \cdots + a_1(x)\bar{y}(x) + a_0(x)y_0(x) = 0 \quad \ldots\ldots..(3)$$

A point x_0 is called an ordinary point of eq.(3) if the coefficient functions $a_i(x)$ are (real) analytic in a neighborhood of x_0, that is, the Taylor series at x_0 converges to the function in a neighborhood of x_0 which means that this D.E. can be solved by power series.

- Remark$_3$: For a homogeneous D.E. that given in equation (4),

$$\bar{\bar{y}} + b(x)\bar{y} + c(x)y = 0 \quad \ldots\ldots..(4)$$

is said to have a regular singular point if $b(x)$ & $c(x)$ are not analytic in a neighborhood of x_0 but when $b(x)$ are multiplied by $(x - x_0)$ and $c(x)$ are multiplied by $(x - x_0)^2$ then these functions will be analytic at x_0 then this point is called a regular singular point.

- Remark$_4$: If one of $[(x - x_0)\, b(x), (x - x_0)^2\, c(x)]$ is not analytic at x_0, this point is said to be irregular singular point, and cannot be solved by power series.

Ex$_3$/ Show if of the following differential equations have ordinary, regular singular and irregular points.

1- $\bar{\bar{y}} + (2 + x)\bar{y} + xy = 0$

2- $\bar{\bar{y}} + e^x\bar{y} + x^{-4}y = 0$

Sol:

1- $\bar{\bar{y}} + (2 + x)\bar{y} + xy = 0$

Since $b(x) = 2 + x$ and $c(x) = x$ are analytic at $x = 0,1,2,3,4, \ldots\ldots\ldots..$ then these points are called ordinary points.

2- $\bar{\bar{y}} + e^x\bar{y} + x^{-4}y = 0$

The first function $b(x) = e^x$ is analytic at $x = 0,1,2,3,4, \ldots\ldots\ldots..$

While the second function $c(x) = \frac{1}{x^4} \to \infty \; as \; x \to 0$, multiply the $b(x) by\; x$ and multiply $c(x)\; by\; x^2 \to \bar{\bar{y}} + xe^x\bar{y} + \frac{1}{x^2}y = 0$

The first function still analytic but the second not analytic, therefore this point is called irregular singular point.

❖ Solving D.E. using Power Series

The power series can be used to find solutions to differential equations of the form of equation (4), since many differential equations can't be solved explicitly in terms of finite combinations of simple familiar functions.

Remark$_5$: the steps of solution of D.E. using power series are:

1- Test each of $b(x)\&c(x)$ if they are analytic or not at x_0

2- If $b(x)\&c(x)$ are analytic at x_0 ,express y in the form of power series

3- Find the first and the second derivatives of y

4- Substitutes the values of (y) and its derivatives in the D.E.

5- Make the power of (x) the same by assuming (n) equal a value of (r) so the value of (r) be equal to the power of (x)

6- Evaluating all coefficients in terms of a_0 & a_1

7- Write (y) in the form of power series with only a_0 & a_1 coefficients

Ex$_4$/ Use power series to solve the equation $\bar{\bar{y}} + y = 0$

Sol:

Since $b(x) = 0$ &$c(x)$ =1then the two functions are analytic and the D.E. can be solved by power series.

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots\dots\dots\dots \sum_{n=0}^{\infty} c_nx^n$$

$$\rightarrow \bar{y} = c_1 + 2c_2x + 3c_3x^2 + \dots\dots\dots\dots \sum_{n=1}^{\infty} nc_nx^{n-1}$$

$$\rightarrow \bar{\bar{y}} = 2c_2 + 6c_3x + 12c_4x^3 + \dots\dots\dots\dots \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}$$

This lead to

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

For the first term let $r = n - 2 \rightarrow n = r + 2$ & $n - 1 = r + 1$ and for the second term let $r = n \rightarrow$

$[(r+2)(r+1)\, c_{r+2} + c_r]x^r = 0$, since $x^r \neq 0$

$(r+2)(r+1)\, c_{r+2} + c_r = 0$

$c_{r+2} = \frac{-c_r}{(r+2)(r+1)}$, this equation is called a recursion relation.

If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting in succession.

$r = 0 \rightarrow c_2 = \frac{-c_0}{1*2}$

$r = 1 \rightarrow c_3 = \frac{-c_1}{2*3}$

$r = 2 \rightarrow c_4 = \frac{-c_2}{3*4} = \frac{c_0}{1*2*3*4} = \frac{c_0}{4!}$

$r = 3 \rightarrow c_5 = \frac{-c_3}{4*5} = \frac{c_1}{2*3*4} = \frac{c_1}{5!}$

$r = 4 \rightarrow c_6 = \frac{-c_4}{5*6} = \frac{-c_0}{4!*5*6} = \frac{-c_0}{6!}$

$r = 5 \rightarrow c_7 = \frac{-c_5}{6*7} = \frac{-c_0}{5!*6*7} = \frac{-c_1}{7!}$

$r = 6 \rightarrow c_8 = \frac{-c_6}{7*8} = \frac{-c_0}{6!*7*8} = \frac{c_0}{8!}$

For even coefficients $c_{2n} = (-1)^n \frac{c_0}{(2n)!}$

For odd coefficients $c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}$

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ … … … …

$= c_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right) + c_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \right)$

It is obvious from the above series is the same as($sinx$ & $cosx$) therefore the function (y) can be written as

$y = c_0 \, sinx + c_1 cosx$

Ex$_5$/ Use Taylor series to find the series solution of

$\bar{y} = 2(y - x)$, if $y = 1$ when $x = 0$.

Sol:

$\bar{y} = 2(y - x) \quad \rightarrow \bar{y}(0) = 2$

$\bar{\bar{y}} = 2y\bar{y} - 2 \quad \rightarrow \bar{\bar{y}}(0) = 2$

$\bar{\bar{\bar{y}}} = 2[y\bar{\bar{y}} + (\bar{y})^2] \quad \rightarrow \bar{\bar{y}}(0) = 12$

And so on to gate

$y = y(0) + \bar{y}(0)x + \bar{\bar{y}}(0)\frac{x^2}{2!} + \bar{\bar{\bar{y}}}(0)\frac{x^3}{3!} + \ldots\ldots$

$y = 1 + 2x + x^2 + 6x^3 + \ldots\ldots$

Ex$_6$/ Solve the following second order D.E.

$\bar{\bar{y}} - 2x\bar{y} + y = 0$, around $x_0 = 0$

Sol:

Since $b(x) = 2x$ & $c(x) = 1$ then the two functions are analytic and the D.E. can be solved by power series.

$$y = \sum_{n=0}^{\infty} a_n \, x^n$$

$$\bar{y} = \sum_{n=1}^{\infty} n a_n \, x^{n-1}$$

$$\bar{\bar{y}} = \sum_{n=2}^{\infty} n(n-1) a_n \, x^{n-2}$$

$\sum_{n=2}^{\infty} n(n-1)c_n\, x^{n-2} - \sum_{n=1}^{\infty} 2nc_n\, x^n + \sum_{n=0}^{\infty} c_n\, x^n = 0$

For the first term $r = n-2 \rightarrow n = r+2 \;\&\; n-1 = r+1$

For the second term $r = n$

For the third term $r = n$

$[(r+2)\,(r+1)\,c_{r+2} - 2rc_r + c_r]\,x^r = 0$

$\rightarrow (r+2)\,(r+1)\,c_{r+2} + (1-2r)c_r = 0$

$\rightarrow c_{r+2} = \frac{(2r-1)}{(r+2)(r+1)}\; c_r$

$r = 0 \;\; \rightarrow \; c_2 = \frac{-1}{2} c_0 = \frac{-1}{2!} c_0$

$r = 1 \;\; \rightarrow \; c_3 = \frac{1}{2*3}\, c_1 = \frac{1}{3!}\, c_1$

$r = 2 \;\; \rightarrow \; c_4 = \frac{3}{4*3}\, c_2 = \frac{-3}{4!} c_0$

$r = 3 \;\; \rightarrow \; c_5 = \frac{5}{4*5}\, c_3 = \frac{5}{5!} c_1$

And so on

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \; ...$

$y = c_0 \left(1 - \frac{1}{2!} x^2 - \frac{3}{4!}\, x^4 + \; ... \right) + c_1 \,(x + \frac{1}{3!} x^3 + \frac{5}{5!}\, x^5 + \cdots)$

Ex$_7$/ Solve the following D.E. using series

$\bar{\bar{y}} + 2\bar{y} + x^2 y = 0$

Sol:

$$y = \sum_{n=0}^{\infty} a_n\, x^n$$

$$\bar{y} = \sum_{n=1}^{\infty} n a_n\, x^{n-1}$$

$$\bar{\bar{y}} = \sum_{n=2}^{\infty} n(n-1) a_n\, x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-2} + 2\sum_{n=1}^{\infty} na_n\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, x^{n+2} = 0$$

For the first term $r = n-2 \rightarrow n = r+2 \;\&\; n-1 = r+1$

For the second term $r = n-1 \rightarrow n = r+1$

For the third term $r = n+2 \rightarrow n = r-2$

$[(r+2)\,(r+1)\, c_{r+2} + 2(r{+}1)c_{r+1} + c_{r-2}]\, x^r = 0$

$\rightarrow (r+2)\,(\mathrm{r}+1)\, c_{r+2} + 2(r{+}1)c_{r+1} + c_{r-2} = 0$

$\rightarrow c_{r+2} = \frac{2}{(r+2)}\, c_{r+1} + \frac{1}{(r+2)(\mathrm{r}+1)}\, c_{r-2}$

$c_2 \& c_3$ can be found as follow:

$r = 2 \;\rightarrow\; c_2 = \dfrac{-1}{2} c_0$

$r = 1 \;\rightarrow\; c_3 = \dfrac{1}{2*3}\, c_{\,1}$

$r = 2 \;\rightarrow\; c_4 = \dfrac{3}{4*3}\, c_{\,2} = \dfrac{-3}{4!} c_0$

$r = 3 \;\rightarrow\; c_5 = \dfrac{5}{4*5}\, c_{\,3} = \dfrac{5}{5!} c_1$

And so on

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ \ldots$

$y = c_0 \left(1 - \frac{1}{2} x^2 - \frac{3}{4!}\, x^4 + \ \ldots\right) + c_{\,1}\,(x + \frac{1}{2*3}\, x^3 + \frac{5}{5!}\, x^5 + \cdots)$

❖ **<u>Frobenius Method</u>**

The method of Frobenius works for differential equations of the form $\bar{\bar{y}}$ + b(x) $\bar{y}$+ c(x) $y = 0$ in which either b(x) & c(x) are not analytic at the point of expansion x_0 or one of them is not analytic at the point of expansion x_0.

To illustrate this method consider the following example.

Ex8/ Solve the following D.E. using Frobenius method

$x^2\bar{\bar{y}} - x\,\bar{y} + (1-x)\,y = 0$

Around $x_0 = 0$

Sol:

The general form of y is:

$y = (x-x_0)^{\lambda} \sum c_n\,(x-x_0)^n \ldots\ldots\ldots(5)$

Since $x_0 = 0$ then

$y = \sum_{n=0}^{\infty} c_n\,x^{n+\lambda}$

$$\rightarrow \bar{y} = \sum_{n=0}^{\infty} c_n\,(n+\lambda)x^{n+\lambda-1}$$

$$\bar{\bar{y}} = \sum_{n=0}^{\infty} c_n\,(n+\lambda)(n+\lambda-1)x^{n+\lambda-2}$$

$$x^2\sum_{n=0}^{\infty} c_n\,(n+\lambda)(n+\lambda-1)x^{n+\lambda-2} - x\sum_{n=0}^{\infty} c_n\,(n+\lambda)x^{n+\lambda-1} + (1-x)\sum_{n=0}^{\infty} c_n\,x^{n+\lambda} = 0$$

$$\sum_{n=0}^{\infty} c_n\,(n+\lambda)(n+\lambda-1)x^{n+\lambda} - \sum_{n=0}^{\infty} c_n\,(n+\lambda)x^{n+\lambda} + (1-x)\sum_{n=0}^{\infty} c_n\,x^{n+\lambda} = 0 \ldots\ldots\ldots(6)$$

- Remark6: the next step of the solution is finding the value of λ by letting $(n=0)$ and make the coefficients of the lowest power equal to zero.

The lowest power when $(n=0)$ is (λ) which mean that:

$[\lambda(\lambda-1)c_0 - \lambda c_0 + c_0]\,x^{\lambda} = 0$ [$x^{\lambda+1}$ is neglected]

$\lambda(\lambda-1)c_0\ -\lambda c_0+\ c_0\ =0$

$\left(\lambda^2-2\lambda+1\right)=0\rightarrow(\lambda-1)^{\ 2}$

$\lambda_1\ =1\ ,\lambda_2\ =1$

Now make all the power of (x) in eq. (6) equal to $(r+\lambda)$ and their coefficients equal to zero.

$\rightarrow[c_r\ (r+\lambda)(r+\lambda-1)-c_r\ (r+\lambda)+c_r-c_{r-1}]\ x^{r+\lambda}=0$

For $\lambda_1\ =1$

$c_r\ [(r+1)(r+1-1)-\ (r+1)+1]-c_{r-1}=0$

$$c_r\ =\frac{c_{r-1}}{r^2}$$

At

$r=1\ \rightarrow\ c_1=\frac{c_0}{(1)^2}$

$r=2\quad\rightarrow\ c_2\ =\frac{c_1}{(2)^2}=\frac{c_0}{(2*1)^2}$

$r=3\quad\rightarrow\ c_3\ =\frac{c_2}{(3)^2}=\ \frac{c_0}{(3*2*1)^2}$

And so on

- Remark$_7$: the solution equations can be written depending on the difference of $(\lambda_1\ and\ \lambda_2)$

When $\lambda_1-\lambda_2\ =0$ the solution will be

$$y_1\ =\sum_{n=0}^{\infty}c_n\ x^{n+\lambda_1}\quad\&\quad y_2\ =y_1(x)\ln(x)+\sum_{n=0}^{\infty}d_n\ x^{n+\lambda_2}$$

When $\lambda_1-\lambda_2\ =integer$ the solution will be

$$y_1\ =\sum_{n=0}^{\infty}c_n\ x^{n+\lambda_1}\quad\&\quad y_2\ =Cy_1(x)\ln(x)+\sum_{n=0}^{+\infty}d_n\ x^{n+\lambda_2}$$

When $\lambda_1-\lambda_2\ =not\ integer$ the solution will be

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+\lambda_1} \quad \& \quad y_2 = \sum_{n=0}^{\infty} c_n\, x^{n+\lambda_2}$$

Since $\lambda_1 - \lambda_2 = 0$ then

$$y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+\lambda_1} \;\rightarrow\; y_1 = \sum_{n=0}^{\infty} c_n\, x^{n+1}$$

$$\rightarrow y_1 = c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + \cdots$$

$$\therefore y_1 = c_0\left(x + \frac{1}{(1)^2}x^2 + \frac{1}{(2*1)^2}x^3 + \frac{1}{(3*2*1)^2}x^4 + \cdots\right)$$

$$y_1 = c_0 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n!)^2}$$

$$y_2 = y_1(x)\ln(x) + \sum_{n=0}^{\infty} d_n\, x^{n+\lambda_2}$$

$$y_2 = y_1(x)\ln(x) + \sum_{n=0}^{\infty} d_n\, x^{n+1}$$

$$\bar{y}_2 = y_1 x^{-1} + \bar{y}_1 \ln(x) + \sum_{n=1}^{\infty} (n+1) d_n\, x^n$$

$$\bar{\bar{y}}_2 = -y_1 x^{-2} + \bar{y}_1 x^{-1} + \bar{y}_1 x^{-1} + \bar{\bar{y}}_1 \ln(x)$$

$$+ \sum_{n=1}^{\infty} n(n+1) d_n\, x^{n-1} = 0$$

Substituting into $[x^2\bar{\bar{y}} - x\,\bar{y} + (1-x)\,y = 0] \rightarrow$

$-y_1 + 2\,\bar{y}_1 x + \bar{\bar{y}}_1 x^2 \ln(x) + \sum_{n=1}^{\infty} n(n+1) d_n\, x^{n+1} - y_1 -$

$x\,\bar{y}_1 \ln(x) - \sum_{n=0}^{\infty} (n+1) d_n\, x^{n+1} + y_1 \ln(x) +$

$\sum_{n=0}^{\infty} d_n\, x^{n+1} - x y_1 ln x - \sum_{n=0}^{\infty} d_n\, x^{n+2} = 0$

$$(x^2\bar{\bar{y}}_1 - 2x\bar{y}_1 + y_1)\ln(x) - 2y_1 + 2x\bar{y}_1 + \sum_{n=1}^{\infty} n(n+1)d_n\, x^{n+1} - \sum_{n=0}^{\infty}(n+1)d_n\, x^{n+1} + \sum_{n=0}^{\infty} d_n\, x^{n+1} - \sum_{n=0}^{\infty} d_n\, x^{n+2} = 0$$

Since y_1 is a solution of the DE, the terms times $(ln\ x)$ above equal 0, and we have:

$$2x\bar{y}_1 - 2y_1 + \sum_{n=1}^{\infty} n(n+1)d_n\, x^{n+1} - \sum_{n=0}^{\infty}(n+1)d_n\, x^{n+1} + \sum_{n=0}^{\infty} d_n\, x^{n+1} - \sum_{n=0}^{\infty} d_n\, x^{n+2} = 0$$

Combining the first three sums and shifting the last one by letting $(n = r-1)$, leads to

$$2x\bar{y}_1 - 2y_1 + \sum_{n=1}^{\infty} n^2 d_n\, x^{n+1} - \sum_{r=2}^{\infty} d_{r-1}\, x^{r+1} = 0$$

$$2x\bar{y}_1 - 2y_1 + d_1x^2 + \sum_{r=2}^{\infty}(r^2 d_r - d_{r-1})x^{r+1} = 0 \ldots\ldots\ldots (7)$$

Since $y_1 = c_0 \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n!)^2} \rightarrow \bar{y}_1 = c_0 \sum_{n=0}^{\infty} \frac{(n+1)x^n}{(n!)^2}$

Thus substituting y_1 and $\bar{y}_1$ into eq.(7):

$$\sum_{n=0}^{\infty}\frac{2(n+1)x^{n+1}}{(n!)^2}-\sum_{n=0}^{\infty}\frac{2x^{n+1}}{(n!)^2}+d_1x^2$$

$$+\sum_{n=1}^{\infty}(r^2d_r-d_{r-1})x^{r+1}=0$$

$$\sum_{n=0}^{\infty}\frac{[2(n+1)-2]x^{n+1}}{(n!)^2}+d_1x^2+\sum_{r=2}^{\infty}(r^2d_r-d_{r-1})x^{r+1}=0$$

$$(2+d_1)x^2+\sum_{r=2}^{\infty}(\frac{2r}{(r!)^2}+r^2d_r-d_{r-1})x^{r+1}=0$$

Now set the coefficients of the powers of x equal to zero.

$2+d_1=0\rightarrow d_1=-2$

$\frac{2r}{(r!)^2}+r^2d_r-d_{r-1}=0$

$$d_r=\frac{1}{r^2}\,[d_{r-1}-\frac{2r}{(r!)^2}]$$

$r=1\rightarrow$

$$d_1=\frac{1}{(1)^2}\left[d_0-\frac{2(1)}{(1!)^2}\right]\rightarrow -2=d_0-2\rightarrow d_0=0$$

And for $r\geq 1\rightarrow$

$d_2=\frac{-3}{4}$, $d_3=\frac{-11}{108}$, and so on

$$\therefore\ \ y_2=y_1(x)\ln(x)+\sum_{n=1}^{\infty}d_n\,x^{n+1}$$

$$=y_1(x)\ln(x)-2x^2-\frac{3}{4}x^3-\frac{11}{108}x^4+\cdots$$