❖ Partial Differential Equation

An equation containing partial derivatives of a function of two or more independent variables is called a partial differential equation (PDE).

If u is a function of (x, y) then:

$(u_x = \frac{du}{dx}), \left(u_y = \frac{du}{dy}\right), \left(u_{xx} = \frac{d^2u}{dx^2}\right), (u_{xy} = \frac{d^2u}{dxy}),$

$and(u_{yy} = \frac{d^2u}{dy^2})$

For example $y^2\frac{du}{dx} + \frac{du}{dy} = u$ can be expressed as

$y^2u_x + u_y = u$

A PDE whose unknown function and its partial derivatives appear linearly in the equation is said to be linear as given in eq. (1)

$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y +$
$F(x,y)u = G(x,y)$ .... (1)

This is the general form of second-order linear PDE. If G(x, y) = zero, then it is said to be homogeneous, otherwise it is non- homogeneous.

The homogenous second order differential equation can be written as:

$Au_{xx} + Bu_{xy} + Cu_{yy} + D(x,y,u(x,y),u_x,u_y) = 0$ .... (2)

Equation (2) can be classified in to three types according to $(B^2 - 4AC)$

1- Elliptic $(B^2 - 4AC < 0)$
2- Parabolic $(B^2 - 4AC = 0)$
3- Hyperbolic $(B^2 - 4AC > 0)$

In general, the solution of PDE presents a much more difficult problem than the solution of ODE and except for certain special types of linear PDE, no general method of solution is available. It is remarkable and fortunate that a large number of the important equations in practice are not only linear, but also of second order, for which solutions are relatively easy to find. The order of the highest derivative is the order of the equation.

Ex$_1$/ Find the solution of the following PDE

$$(x-2)\frac{du}{dx} - y\frac{du}{dy} = 0$$

Sol:

Assume that the solution is $u = g(x).f(y)$, where (g) is a function of (x) is only and (f) is a function of (y) only.

This mean that when derive u with respect to x assume y constant and when derive u with respect to y assume x constant.

$\rightarrow \frac{du}{dx} = \bar{g}\, f$, And $\frac{du}{dy} = \bar{f}\, g$

This means that

$(x-2)\bar{g}\, f - y\bar{f}\, g = 0$

$(x-2)\frac{\bar{g}}{g} = y\frac{\bar{f}}{f}$, since each side of this equation is function of one variable only then

$$(x-2)\frac{\bar{g}}{g} = y\frac{\bar{f}}{f} = k$$

$\rightarrow\ (x-2)\frac{\bar{g}}{g} = k \rightarrow \frac{\bar{g}}{g} = \frac{k}{(x-2)} \rightarrow \int\frac{\bar{g}}{g} = \int\frac{k}{x-2}\, dx$

$\therefore \ln g = k\ln(x-2) + c_1$

Let $c_1 = \ln c_2$ where $c_2$is constant too $\rightarrow \ln g = \ln(x-2)^k + lnc_2$

$\rightarrow \ln g = \ln c_2(x-2)^k$, taking exponential for two sides then:

$g\ = c_2\ (x-2)^k$

In the same way:

$y\frac{\bar{f}}{f} = k\ \rightarrow \frac{\bar{f}}{f} = \frac{k}{y}$, taking integration with respect to (y) for two sides then

$\int \frac{\bar{f}}{f} = \int \frac{k}{y} \rightarrow \ln f = k ln\ y + c_2$, let $c_2 = \ln c_3$

$\ln f = ln\ y^k + \ln c_3$

$\ln f = \ln c_3 y^k$

$f = c_3\ (y)^k$

$\therefore u = c_2\ (x-2)^k c_3\ (y)^k = \mathrm{C}\ (x-2)^k (y)^k\ [C = c_2 c_3]$

$\rightarrow u = C\ (xy - 2y)^k$

HW_1: Validate the solution of Ex_1

There are some examples of PDE which are:

1- $u_t = c^2 u_{xx}$ one dimensional Heat conduction equation

2- $u_{tt} = c^2 u_{xx}$ one dimensional Wave equation

3- $u_{xx} + u_{yy} = 0$ two dimensional Laplace equation

4- $u_{xx} + u_{yy} = f(x, y)$ two dimensional Position equation

5- $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt} - \lambda^2$ u two dimensional Klein-Gordon equation

6- $u_{tt} = C^2 (u_{xx} + u_{yy} + u_{zz}$ three dimensional Wave equation

Ex_2/ Solve the following partial differential equation

$$\frac{d^2u}{dx} - 2xdy = 0$$

Sol:

$$\frac{d^2u}{dx} = 2xdy$$

$$\frac{d^2u}{dxdy} = 2x$$

First: integrate with respect to (x)

$$\frac{du}{dy} = x^2 + f(y)$$

Second: integrate with respect to y

$$u = x^2 y + f(y) + g(x)$$

Ex$_3$/ Solve the following partial differential equation

$$\frac{d^2y}{dtdx} = cos^2(x) - t$$

$$cos^2x + sin^2x = 1$$
$$cos2x = cos^2x - sin^2x$$
$$cos2x = cos^2x - 1 + cos^2x$$
$$cos^2x = \frac{cos2x + 1}{2}$$

Sol:

Integration with respect to (x)

$$\frac{dy}{dt} = \frac{1}{2}[\frac{1}{2}(sin(2x) + x] - tx + f(t)$$

Integration with respect to (t)

$$y = \frac{t}{2}\ [\frac{1}{2}(sin(2x) + x] - \frac{xt^2}{2} + f(t) + g(x)$$

Ex$_4$/ For the following conditions, solve the partial differential equation

$$w_{xy} - 2\cos(y) = e^{2x} - y$$

$$w_y = 2y \quad at\ x = 0\ \ and\ w = 3e^x \quad at\ y = 0$$

Sol:

$$\frac{d^2w}{dxdy} - 2cosy = e^{2x} - y$$

Integration with respect to (x)

$$\frac{dw}{dy} - 2xcosy = \frac{1}{2}e^{2x} - yx + f(y)$$

When $x = 0\ then \frac{dw}{dy} = 2y$

$$\rightarrow 2y = \frac{1}{2} + f(y)$$

$$f(y) = 2y - \frac{1}{2}$$

Integration with respect to (y)

$$w - 2xsiny = \frac{y}{2}e^{2x} - \frac{1}{2}y^2x + y^2 - \frac{1}{2}y + g(x)$$

When $y = 0\ \ then\ w = 3e^x$

$$3e^x = g(x)$$

$$\therefore w(x,y) = 2xsiny + \frac{y}{2}e^{2x} - \frac{1}{2}y^2x + y^2 - \frac{1}{2}y + 3e^x$$

❖ **Solution of one Dimensional Heat Equation**

Consider a thin bar of length L, of uniform cross-section and constructed of homogeneous material l. Suppose that the side of the bar is perfectly insulated so no heat transfer could occur through it (heat could possibly still move into or out of the bar through the two ends of the bar). Thus, the movement of heat inside the bar could occur only in the x-direction. Then, the amount of heat content at any place inside the bar: $[0 < x < L]$, and at any time $t > 0$, is given by the temperature distribution function $u(x, t)$. It satisfies the homogeneous one-dimensional heat conduction equation:

$\alpha^2\, u_{xx} = u_t$ …….. (3)

Where the constant coefficient (α^2) is the thermo diffusivity of the bar, which is given by ($\alpha^2 = \frac{k}{\rho s}$). (k = thermal conductivity, ρ = density, and s = specific heat of the material of the bar.)

To solve this equation the following conditions must be considered:

1- Boundary condition

$u(0,t) = 0$ for $t \geq 0$& $u(L,t) = 0$ $t \geq 0$

2- Initial condition

$u(x,0) = f(x) for\ 0 < x < L$

A major difference now is that the general solution is dependent not only on the equation, but also on the boundary conditions, while the particular solution depends on the initial condition too.

Let the solution of this equation is

$u = XT$

Then:

$u_x = \bar{X}T,\ u_t = \bar{T}X,\ u_{xx} = \bar{\bar{X}}\,T,\ \ u_{tt} = X\bar{\bar{T}},\ \ u_{xt} = \bar{X}\bar{T}$

This means that equation three can be written as

$\alpha^2\ \bar{\bar{X}}\,T = \bar{T}X$, dividing on (α^2) then

$\bar{\bar{X}}\,T = \frac{1}{\alpha^2}\bar{T}X$ …….. (4)

Equation (4) can be written as:

$$\frac{\bar{\bar{X}}}{X} = \frac{1}{\alpha^2}\frac{\bar{T}}{T}$$

The critical idea here is that, because the independent variables x and t are vary independently, in order for the above equation to hold for all values of (x) and (t), the expressions on both sides of the equation must be equal to the same constant. Let us call the constant (k). It is called the constant of separation. Thus:

$$\frac{\bar{\bar{X}}}{X} = \frac{1}{\alpha^2}\frac{\bar{T}}{T} = k$$

This means that:

$$\frac{\bar{\bar{X}}}{X} = k \rightarrow \bar{\bar{x}} = kx$$

$\bar{\bar{X}} - kX = 0$ …….. (5)

$$\frac{1}{\alpha^2}\frac{\bar{T}}{T} = k \rightarrow \bar{T} = \alpha^2 Tk$$

$\bar{T} - \alpha^2 kT = 0$ …….. (6)

There are three cases of solution depending on the value of k

- When $k = \lambda^2$ then eq.(5) & eq.(6) can be written as:

$\bar{\bar{X}} - \lambda^2 X = 0$ …….. (7)

$\bar{T} - \alpha^2\lambda^2 T = 0$ …….. (8)

The solutions of eq. (7) & eq. (8) is

$X(x) = A_1 e^{\lambda x} + B_1\, e^{-\lambda x}$ & $T(t) = C_1 e^{\alpha^2\lambda^2 t}$

- When $k = -\lambda^2$ then eq.(5) & eq.(6) can be written as:

$\bar{\bar{X}} + \lambda^2 X = 0$ ....(9)

$\bar{T} + \alpha^2 \lambda^2 T = 0$ ....(10)

The solutions of eq. (9) & eq. (10) is

$X(x) = A_2 \cos \lambda x + B_2 \sin \lambda x$ & $T(t) = C_2 e^{-\alpha^2 \lambda^2 t}$

- When $k = 0$ then eq.(5) & eq.(6) can be written as

$\bar{\bar{X}} = 0$ ....(11)

$\bar{T} = 0$ ....(12)

The solutions of eq. (9) & eq. (10) is

$X(x) = A_3 x + B_3$ & $T(t) = C_3$

$\rightarrow$

$$u = \left\{ \begin{array}{ll} (A_1 e^{\lambda x} + B_1 e^{-\lambda x}) * C_1 e^{\alpha^2 \lambda^2 t} & for\ k = \lambda^2 \\ (A_2 \cos \lambda x + B_2 \sin \lambda x)\, C_2 e^{-\alpha^2 \lambda^2 t} & for\ k = -\lambda^2 \\ (A_3 x + B_3) * C_3 & for\ k = 0 \end{array} \right\}$$

The second solution of u is considered since this solution represents the periodic one,

For the boundary condition $u(0,t) = 0$

$(A_2 \cos \lambda x + B_2 \sin \lambda x)\, C_2 e^{-\alpha^2 \lambda^2 t} = 0$

$(A_2 \cos(\lambda * 0) + B_2 \sin (\lambda * 0))\, C_2 e^{-\alpha^2 \lambda^2 t} = 0$

$A_2 C_2 e^{-\alpha^2 \lambda^2 t} = 0 \rightarrow A_2 = 0$

For the boundary condition $u(L,t) = 0$

$(0 * \cos \lambda L + B_2 \sin \lambda L) C_2 e^{-\alpha^2 \lambda^2 t} = 0$

$B_2 \sin(\lambda L)\, C_2 e^{-\alpha^2 \lambda^2 t} = 0$

$\rightarrow B_2 \neq 0$ & $\sin(\lambda L) = 0 \rightarrow \lambda L = n\pi$

$\lambda = \frac{n\pi}{L}$. Where n is integer.

$u(x,t) = B_2 \sin\left(\frac{n\pi}{L} x\right) C_2 e^{\frac{-\alpha^2 n^2 \pi^2 t}{L^2}}$, let $B_n = B_2\, C_2$

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-\alpha^2 n^2 \pi^2 t}{L^2}}$$

For initial condition

$$u(x,0) = f(x) for \ 0 < x < L$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

Using Fourier sine series expansion: $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right) . B_n \sin\left(\frac{n\pi}{L}x\right) e^{\frac{-\alpha^2 n^2 \pi^2 t}{L^2}}$$

- **Solution of wave Equation**

The separation of variables technique is used to study the wave equation on a finite interval. To illustrate the physical origin of the wave equation consider small transverse (one dimensional) vibrations of an elastic string with ends fixed at $x = 0$ and $x = L$. The general form of the speed with which wave is propagated is given in equation:

$$\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2} \quad \ldots\ldots..(13)$$

To solve this equation, separation of variables can be used as follow:

First step: Consider the boundary and initial conditions:

$$\left.\begin{aligned} &y(0,t) = 0 \ for\ t \geq 0 \\ &y(L,t) = 0 \ for\ t \geq 0 \end{aligned}\right\} \text{Boundary conditions}$$

$$\left.\begin{aligned} &y(x,0) = f(x) \ for\ 0 < x < L \\ &\left(\frac{dy}{dt}\right)_{t=0} = g(x) \ for\ 0 < x < L \end{aligned}\right\} \text{Initial conditions}$$

Second step: Finding the factorized solutions

The factorized function $y(x,t) = X(x)T(t)$ is a solution to the wave equation (13) if and only if

$\frac{dy}{dt} = X\bar{T}, \frac{d^2y}{dt^2} = X\bar{\bar{T}}$
$\frac{dy}{dx} = T\bar{X}, \frac{d^2y}{dX^2} = T\bar{\bar{X}}$

$X(x)\bar{\bar{T}}(t) = c^2\bar{\bar{X}}(x)T(t)$ Or $\frac{\bar{\bar{X}}}{X} = \frac{\bar{\bar{T}}}{c^2T}$

$\rightarrow \frac{\bar{\bar{X}}}{X} = \frac{\bar{\bar{T}}}{c^2T} = k$, Where k is constant

$\frac{\bar{\bar{X}}}{X} = k \rightarrow \bar{\bar{X}} = kX$

$\bar{\bar{X}} - kX = 0 \quad \ldots\ldots(14)$

$\frac{\bar{\bar{T}}}{c^2T} = k \rightarrow \bar{\bar{T}} = c^2Tk \rightarrow$

$\bar{\bar{T}} - c^2kT = 0 \quad \ldots\ldots(15)$

- When $k = \lambda^2$ then eq.(14) & eq.(15) can be written as:

$\bar{\bar{X}} - \lambda^2X = 0 \quad \ldots\ldots(16)$

$\bar{\bar{T}} - c^2\lambda^2T = 0 \quad \ldots\ldots(17)$

The solutions of eq. (16) & eq. (17) is

$X(x) = A_1e^{\lambda x} + B_1e^{-\lambda x} \quad \& \quad T(t) = C_1e^{c\lambda t} + D_1e^{-c\lambda t}$

- When $k = -\lambda^2$ then eq.(14) & eq.(15) can be written as:

$\bar{\bar{X}} + \lambda^2X = 0 \quad \ldots\ldots(18)$

$\bar{\bar{T}} + c^2\lambda^2T = 0 \quad \ldots\ldots(19)$

The solutions of eq. (18) & eq. (19) is

$X(x) = A_2\cos\lambda x + B_2\sin\lambda x \quad \& \quad T(t) = C_2\cos c\lambda t + D_2\sin c\lambda t$

- When $k = 0$ then eq.(14) & eq.(15) can be written as

$\bar{\bar{X}} = 0 \quad \ldots\ldots(20)$

$\bar{\bar{T}} = 0 \quad \ldots\ldots(21)$

The solutions of eq. (20) & eq. (21) is

$X(x) = A_3x + B_3 \quad \& \quad T(t) = C_3t + D_3$

$$\rightarrow y = \left\{ \begin{array}{c} (A_1e^{\lambda x} + B_1e^{-\lambda x})(C_1e^{c\lambda t} + D_1e^{-c\lambda t}) \ \ for\ k = \lambda^2 \\ (A_2\cos\lambda x + B_2\sin\lambda x)(C_2\cos c\lambda t + D_2\sin c\lambda t)\ for\ k = -\lambda^2 \\ (A_3x + B_3)\ (C_3t + D_3) \qquad for\ k = 0 \end{array} \right\}$$

Step three: find the periodic solution and constants $(A_2, B_2, C_2, and\ D_2)$

The second solution of y is considered since this solution represents the periodic solution and using the boundary conditions:

For $y(0,t) = 0\ for\ t \geq 0$

$(A_2 \cos\lambda(0) + B_2 \sin\lambda(0))(C_2 \cos c\lambda t + D_2 \sin c\lambda t) = 0$

$\rightarrow A_2(C_2 \cos c\lambda t + D_2 \sin c\lambda t) = 0$

Since $(C_2 \cos c\lambda t + D_2 \sin c\lambda t) \neq 0$ then $A_2 = 0$

For $y(L,t) = 0\ for\ t \geq 0$

$((0)\cos\lambda(L) + B_2 \sin\lambda(L))(C_2 \cos c\lambda t + D_2 \sin c\lambda t) = 0$

$\rightarrow (B_2 \sin\lambda L)(C_2 \cos c\lambda t + D_2 \sin c\lambda t) = 0$

$\rightarrow (B_2 \sin\lambda L) = 0 \rightarrow B_2 \neq 0\ \&\ \sin\lambda L = 0 \rightarrow \lambda L = n\pi\ \therefore\ \lambda = \frac{n\pi}{L}$

The most general solution by assuming $[B_2 C_2 = C_n\ \& B_2 D_2 = D_n]$ is

$$y(x,t) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi}{L}x\right)\left(C_n \cos c\frac{n\pi}{L}t + D_n \sin c\frac{n\pi}{L}t\right) \quad \ldots\ldots..(22)$$

Using the initial condition to find $(C_2 and\ D_2)$

For $y(x,0) = f(x)$, eq.(22) will be

$$y(x,0) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi}{L}x\right)\left(C_n \cos c\frac{n\pi}{L}0 + D_n \sin c\frac{n\pi}{L}0\right) = f(x)$$

$\rightarrow$

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x = f(x)$$

For $\left(\frac{dy}{dt}\right)_{t=0} = g(x)$

$$\left(\frac{dy}{dt}\right)_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\frac{n\pi}{L}x = g(x)$$

Using Fourier sine series expansion:

$$C_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx$$

$$\frac{n\pi c}{L}D_n = \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi}{L}x)\,dx$$

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty}(\sin\frac{n\pi}{L}\ x)\ ([\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx]\cos c\frac{n\pi}{L}\ t$$

$$+ [\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx]sinc\frac{n\pi}{L}\ t)\ \ldots\ldots.(23)$$

Ex$_5$/ Solve the following partial differential equation $\frac{d^2u}{dt^2} = c^2\frac{d^2u}{dx^2}$ where u is a function of (t and x) if the boundary & initial conditions are

$u(0,t) = 0\ for\ t \geq 0\ ,\ u(1,t) = 0\ for\ t \geq 0$

$u(x,0) = x(1-x)\ for\ 0 < x < L, (\frac{du}{dt})_{t=0} = 0\ for\ 0 < x < L$

Sol:

$C_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx$

$= \frac{2}{1}\int_0^1 x(1-x)\sin(\frac{n\pi}{1}x)\,dx$

$= [\frac{x^2-x}{n\pi}\cos(n\pi x) + \frac{1-2x}{n^2\pi^2}\sin(n\pi x) - \frac{2}{n^3\pi^3}\cos(n\pi x)]_0^1$

$(x-x^2)$	$\sin(n\pi x)$
$1-2x$	$+\ \frac{-1}{n\pi}\cos(n\pi x)$
-2	$-\ \frac{-1}{n^2\pi^2}\sin(n\pi x)$
0	$+\ \frac{1}{n^3\pi^3}\cos(n\pi x)$

$$\rightarrow C_n = \begin{cases}\frac{8}{n^3\pi^3} & for\ odd\ (n) \\ 0 & for\ even\ (n)\end{cases}$$

Since $g(x) = 0$ (from the initial condition) then

$$u(x,t) = \sum_{n=1}^{\infty}(\sin n\pi)\ (C_n\cos cn\pi\ t\ )$$

Where (c) is constant and (n) is odd and integer.

HW$_2$/ repeat example (5) if $f(x) = sin5x + 2sin7x$ and $g(x) = 3x$

❖ Solution of Laplace Equation

This partial differential equation represents the steady state of a field that depends on two or more independent variables, which are typically spatial. The two dimensions Laplace equation is given in equation (24)

$$\frac{d^2u}{dx^2} = -\frac{d^2u}{dy^2} \quad \dots \dots .(24)$$

In addition, three dimensions Laplace equation is given in equation (25)

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0 \quad \dots \dots .(25)$$

Note that the equations (24, 25) have no dependence on time, just on the spatial variables (x, y). This means that Laplace's Equation describes steady state situations such as:

- Steady state temperature distributions
- Steady state stress distributions
- Steady state potential distributions (it is also called the potential equation
- Steady state flows.

To separation of variables can be used to solve Laplace equation as follow:

First step: Consider the boundary and initial conditions:

$u(0, y) = 0 \; for \; 0 \leq \; y \leq \infty$

$u(L, y) = 0 \; for \; 0 \leq \; y \leq \infty$

$u(x, \infty) = 0 \; for \; 0 \leq x \leq L$

$u(x, 0) = f(x) \; for \; 0 < x < L$

Second step: Finding the factorized solutions

The factorized function $u(x, t) = X(x)T(t)$ is a solution to the wave equation (24) if and only if

$$\frac{\bar{\bar{X}}}{X} = -\frac{\bar{\bar{Y}}}{Y}$$

$$\rightarrow \frac{\bar{\bar{X}}}{X} = -\frac{\bar{\bar{Y}}}{Y} = k$$

Where k is constant and X is a function of (x), Y is a function of (y) only and (k) is constant.

$$\bar{\bar{X}} - kX = 0 \quad \ldots\ldots(26)$$

$$\bar{\bar{Y}} + kY = 0 \quad \ldots\ldots(27)$$

- When $k = \lambda^2$ then eq.(26) & eq.(27) can be written as:

$$\bar{\bar{X}} - \lambda^2 X = 0 \quad \ldots\ldots(28)$$

$$\bar{\bar{Y}} + \lambda^2 Y = 0 \quad \ldots\ldots(29)$$

The solutions of eq. (28) & eq. (29) is

$$X(x) = A_1 e^{\lambda x} + B_1 e^{-\lambda x} \quad \& \quad Y(y) = C_1 \cos\lambda y + D_1 \sin\lambda y$$

- When $k = -\lambda^2$ then eq.(26) & eq.(27) can be written as:

$$\bar{\bar{X}} + \lambda^2 X = 0 \quad \ldots\ldots(30)$$

$$\bar{\bar{Y}} - \lambda^2 Y = 0 \quad \ldots\ldots(31)$$

The solutions of eq. (30) & eq. (31) is

$$X(x) = A_2 \cos\lambda x + B_2 \sin\lambda x \quad \& \quad Y(y) = C_2 e^{\lambda y} + D_2 e^{-\lambda y}$$

- When $k = 0$ then eq.(26) & eq.(24)can be written as

$$\bar{\bar{X}} = 0 \quad \ldots\ldots(32)$$

$$\bar{\bar{Y}} = 0 \quad \ldots\ldots(33)$$

The solutions of eq. (32) & eq. (33) is

$$X(x) = A_3 x + B_3 \quad \& \quad Y(y) = C_3 y + D_3$$

$$\rightarrow y = \left\{ \begin{array}{ll} (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 \cos\lambda y + D_1 \sin\lambda y) & for\ k = \lambda^2 \\ (A_2 \cos\lambda x + B_2 \sin\lambda x)(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) & for\ k = -\lambda^2 \\ (A_3 x + B_3)\ (C_3 y + D_3) & for\ k = 0 \end{array} \right\}$$

Step three: find the constants $(A_2, B_2, C_2, and\ D_2)$

For the second solution and use the boundary condition $u(0,y) = 0\ for\ 0 \leq y \leq \infty$

$(A_2 \cos\lambda(0) + B_2 \sin\lambda(0))(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) = 0$

$\rightarrow A_2(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) = 0$

Since $(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) \neq 0$ then $A_2 = 0$

For $u(L, y) = 0 \; for \; 0 \leq y \leq \infty$

$((0)\cos\lambda(L) + B_2 \sin\lambda(L))(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) = 0$

$\rightarrow (\mathrm{B_2 \sin\lambda L})(C_2 e^{\lambda y} + D_2 e^{-\lambda y}) = 0$

$\rightarrow (\mathrm{B_2 \sin\lambda L}) = 0 \rightarrow \mathrm{B_2} \neq 0 \;\&\; \mathrm{\sin\lambda L} = 0 \rightarrow \mathrm{\lambda L} = n\pi \;\therefore\; \lambda = \frac{n\pi}{\mathrm{L}}$

Using the condition $u(x, \infty) = 0 \; for \; 0 \leq x \leq L$

$(\mathrm{B_2 \sin\lambda x})(C_2 e^{\lambda(\infty)} + D_2 e^{-\lambda(\infty)}) = 0$, since $e^{-\lambda(\infty)}=0$

$\rightarrow (\mathrm{B_2 \sin\lambda x})\left(C_2 e^{\lambda(\infty)}\right) = 0$

But $\mathrm{B_2} \sin(\lambda \mathrm{x}) \left(e^{\lambda(\infty)}\right) \neq 0$

This means that $C_2 = 0$

The most general solution by assuming $[\mathrm{B_2} = \mathrm{B_n}]$ is

$$u(x, y) = \sum_{n=1}^{\infty} \mathrm{B_n}(\sin\frac{n\pi}{\mathrm{L}}\mathrm{x}) \; e^{\frac{n\pi}{\mathrm{L}}y} \quad \ldots\ldots..(34)$$

To find the suitable expression of $(\mathrm{B_n})$, the condition

$u(x, 0) = f(x) \; for \; 0 < x < L$

Equation (31) will be

$$u(x, 0) = \sum_{n=1}^{\infty} \mathrm{B_n}(\sin\frac{n\pi}{\mathrm{L}}\mathrm{x}) = f(x)$$

Using Fourier sine series expansion: $\mathrm{B_n} = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{\mathrm{L}}x)\,dx$

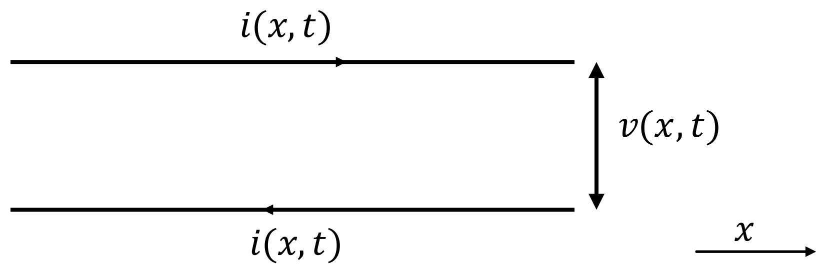
$\rightarrow$

$$u(x, y) = \sum_{n=1}^{\infty} (\frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{\mathrm{L}}x)\,dx)(\sin\frac{n\pi}{\mathrm{L}}\mathrm{x}) \; e^{\frac{n\pi}{\mathrm{L}}y}$$

❖ Other important PDEs in science and engineering

1- Transmission Line Equations

In a long electrical cable or a telephone wire, both the current and voltage depend upon position along the wire as well as the time as illustrated in the following figure.



It is possible to show, using basic laws of electrical circuit theory, that the electrical current $i(x,t)$ satisfies the PDE:

$$\frac{d^2i}{dx^2} = LC\,\frac{d^2i}{dt^2} + (RC + GL)\frac{di}{dt} + RGi \quad \ldots\ldots..(35)$$

Where the constants (R, L, C and G) are, for unit length of cable, respectively the resistance, inductance, capacitance and leakage conductance. The voltage $v(x,t)$ also satisfies eq. (35). Special cases of eq. (35) arise in particular situations. For a submarine cable, G is negligible and frequencies are low so inductive effects can also be neglected. In this case, eq. (35) becomes

$$\frac{d^2i}{dx^2} = RC\,\frac{di}{dt} \ldots\ldots..(35)$$

Which is called the submarine equation or telegraph equation. For high frequency alternating currents, again with negligible leakage, eq. (35) can be approximated by

$$\frac{d^2i}{dx^2} = LC\,\frac{d^2i}{dt^2} \quad \ldots\ldots..(36)$$

Which is called the high frequency line equation.

2- Poisson's equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f(x,y) \quad \dots\dots..(37)$$

Where f(x, y) is a given function. This equation arises in electrostatics, elasticity theory and elsewhere.

3- Helmholtz's equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + ku^2 = 0 \quad \dots\dots..(38)$$

Is a two dimensional form which arises in wave theory.

4- Schrodinger's equation

$$\left(\frac{-h^2}{8\pi^2 m}\right)[\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = E\varphi \quad \dots\dots..(39)$$

Which arises in quantum mechanics, where (h) is Planck's constant.)

5- Transverse vibrations in a homogeneous rod

$$a^2\frac{d^4u}{dx^4} + \frac{d^2u}{dt^2} = 0 \quad \dots\dots..(40)$$

Where $u(x, t)$ is the displacement at time t of the cross-section through (x).