❖ 1Gamma Function

Let (x) be a real number that is not zero or a negative integer. Then the gamma function, denoted by (Γ), is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \ldots\ldots\ldots(1)$$

Note that the above integral is not defined for all values of x. In fact we can only say that $\Gamma(x)$ is defined everywhere except at (0) and for negative integers.

❖ Properties of gamma function

1- $\Gamma(x+1) = x\Gamma(x)$

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Use integration by parts to obtain

Let $u = e^{-t} \rightarrow du = -e^{-t}$

And $dV = t^{x-1} \rightarrow V = \frac{t^x}{x}$

$$= [\frac{t^x}{x}e^{-t}]_0^\infty + \int_0^\infty \frac{t^x}{x}\, e^{-t}dt$$

$$= 0 + \frac{1}{x}\int_0^\infty t^x e^{-t}dt, \text{ but } \int_0^\infty t^x e^{-t}dt = \Gamma(x+1)$$

$= \frac{1}{x}\,\Gamma(x+1)$ this means that $\Gamma(x) = \frac{1}{x}\,\Gamma(x+1)$

$\therefore \Gamma(x+1) = x\,\Gamma(x)$

2- $\Gamma(x) = (x-1)!$

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Since

$\Gamma(x+1) = x\,\Gamma(x)$ Then

$\Gamma(1) = (0)!\,, \Gamma(2) = (1)!, \Gamma(3) = (2)!\,, \ldots\ldots.$ etc

First $\Gamma(1)$ must be found:

$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} dt$

$$= \int_0^\infty e^{-t} dt$$

=1

3- The Gamma function Γ is differentiable for all $x > 0$

$\bar{\Gamma}(x) = \int_0^\infty \ln(t) t^{x-1} e^{-t} dt$

4- $\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{sin\, x\pi}$

5- $\Gamma\frac{1}{2} = \sqrt{\pi}$

6- $\frac{-1}{x}\Gamma(1-x) = \Gamma(-x)$

Ex$_1$/ Evaluate $\Gamma(8)$

Sol:

Since $\Gamma(x) = (x-1)!$

Then $\Gamma(8) = 7!$

= 5040

Ex$_2$/ Evaluate $\frac{\Gamma(6)}{2\Gamma(3)}$

Sol:

$$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5 * 4 * 3\Gamma(3)}{2\Gamma(3)}$$

= 30

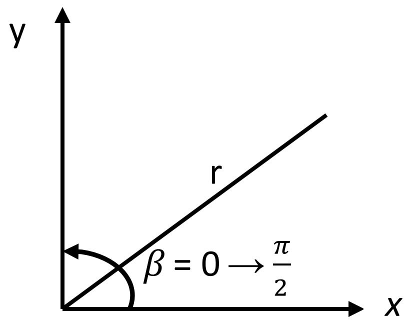
Ex$_3$/ Use the principle of Gamma function to evaluate the following integral $f = \int_0^\infty e^{-x^2}\, dx$

Sol: let $f = \int_0^\infty e^{-x^2}\, dx$, and $f = \int_0^\infty e^{-y^2}\, dy$

This means that $f^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\, dx\, dy$

$$f^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

To solve this dabble integral consider the following figure:



From the previous figure and according to Pythagoras theory

$r^2 = x^2 + y^2$

And $x = r\cos\beta$, $y = r\sin\beta$

$dxdy = rdrd\beta$

$\rightarrow$

$f^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, rdr\, d\beta$

$= [\beta]_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, rdr \longrightarrow$

$$\int_0^\infty e^{-r^2}\, rdr = \frac{-1}{2}\int_0^\infty -2e^{-r^2}\, rdr$$

$= \frac{-1}{2}\left[e^{-r^2}\right]_0^\infty$

$= \frac{1}{2}\left[e^{-r^2}\right]_\infty^0$

$= \frac{1}{2}[1-0]$

$= \frac{1}{2}$

$\therefore\ f^2 = \frac{\pi}{4} \rightarrow f = \frac{\sqrt{\pi}}{2}$

Ex$_3$/ Find $\Gamma\left(\frac{3}{2}\right)$

Sol:

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

$\rightarrow \Gamma\left(\frac{3}{2}\right) = \int_0^\infty t^{\frac{3}{2}-1} e^{-t} dt$

$$= \int_0^\infty t^{\frac{1}{2}} e^{-t} dt$$

Let $t = x^2 \rightarrow dt = 2xdx$

$\Gamma\left(\frac{3}{2}\right) = \int_0^\infty 2\,x^2 e^{-x^2}\,dx$

$\rightarrow \Gamma\left(\frac{3}{2}\right) = [xe^{-x^2}]_2^\infty + \frac{\sqrt{\pi}}{2}$

$\therefore \Gamma\left(\frac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$

$$\begin{array}{ccc} x & & 2xe^{-x^2} \\ & \searrow^{+} & \\ 1 & & -\,e^{-x^2} \\ & \searrow^{-} & \\ 0 & & -\dfrac{\sqrt{\pi}}{2} \end{array}$$

In the same way, $\Gamma\left(\frac{1}{2}\right)$ can be found as ($\sqrt{\pi}$)

Ex$_5$/ Evaluate $\int_0^\infty x^3 e^{-x}\,dx$

Sol:

By compression with the general Gamma equation

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

Then we find that

$$\Gamma(4) = \int_0^\infty x^{4-1} e^{-x} dx$$

$= \int_0^\infty x^3 e^{-x} dx$

$= 3!$

Ex$_6$/ Find $\int_0^1 \frac{dx}{\sqrt{-\ln(x)}}$

Sol:

Let $y = -\ln(x) \rightarrow e^{-y} = x$ and $dx = -e^{-y}$

As $x \rightarrow 0$ then $y \rightarrow \infty$ and as $x \rightarrow 1$then $y \rightarrow \infty$

$\rightarrow \int_0^1 \frac{dx}{\sqrt{-\ln(x)}} = \int_0^\infty \frac{e^{-y}}{\sqrt{y}}\,dy$

$= \int_0^\infty y^{\frac{-1}{2}} e^{-y}\,dy$ Compare with $\int_0^\infty x^{\frac{1}{2}-1} e^{-x} dx$

$= \Gamma\left(\frac{1}{2}\right)$

$=\sqrt{\pi}$ (prove).

Beta Function

The beta function is a two-parameter composition of gamma functions that has been useful enough in application to gain its own name. Its definition is:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad \ldots\ldots\ldots(2)$$

If $(x \geq 1\ and\ y \geq 1)$ this is a proper integral. If $(x > 0;\ y > 0)$ and either or both $(x < 1\ or\ y < 1)$, the integral is improper but convergent. Beta function can be expressed through gamma functions in the following way

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Many integrals can be expressed through beta and gamma functions. Two of special interest are:

$$\int_0^{\frac{\pi}{2}} sin^{2x-1}(\theta)\cos^{2y-1}(\theta)\,d\theta = \frac{1}{2}\,\beta(x,y) = \frac{1}{2}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\,dx = \Gamma(p)\Gamma(p-1) = \frac{\pi}{sin\pi p} \qquad 0 < p < 1$$

Ex$_7$/ Prove that $\beta(x,y) = \beta(y,x)$

Sol:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Let $t = 1-u \rightarrow dt = -du$

As $t \rightarrow 0$ then $u \rightarrow 1$and as $t \rightarrow 1$ then $u \rightarrow 0$

$$\rightarrow \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^1 u^{y-1}(1-u)^{x-1}\,du$$

$$\therefore\ \beta(x,y) = \beta(y,x)$$

Ex$_8$/ Prove that $\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$

Sol:

$$\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dx$$

Let $t = x^2 \rightarrow dt = 2xdx$

$\rightarrow \Gamma(u) = \int_0^\infty x^{2u}x^{-2}(2x)e^{-t}dx$

$= 2\int_0^\infty x^{2u-1}e^{-x^2}dx$

And in the same way

$$\Gamma(v) = 2\int_0^\infty y^{2v-1}e^{-y^2}dy$$

Transforming to polar coordinates [$x = \rho cos\emptyset$, y = $\rho sin\emptyset$]

$$\Gamma(u)\Gamma(v) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty \rho^{2(u+v)-1}e^{-\rho^2}cos^{2u-1}(\emptyset)sin^{2v-1}(\emptyset)d\rho d\emptyset$$

$= 4[\int_0^\infty \rho^{2(u+v)-1}e^{-\rho^2}d\rho]\ [\int_0^{\frac{\pi}{2}} cos^{2u-1}(\emptyset)sin^{2v-1}(\emptyset)d\emptyset]$

$= 2\,\Gamma(u+v)\beta(u,v)$

$$\therefore\ \beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

Ex$_9$/ Evaluate $\int_0^1 x^4(1-x)^3dx$

Sol:

$$\int_0^1 x^4(1-x)^3dx = \beta(5,4)$$

$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)}$

$= \frac{4!*3!}{8!}$

Ex$_{10}$/ Show that $\int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)\cos^{2v-1}(\theta)d\theta = \frac{\Gamma(u)\Gamma(v)}{2\Gamma(u+v)}$

Sol:

Since $\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ then $\beta(u,v)$ must be equal to $(2\int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)\cos^{2v-1}(\theta)d\theta)$

$$\beta(u,v) = \int_0^1 x^{u-1}\,(1-x)^{v-1}dx$$

Let $x = sin^2(\theta) \rightarrow dx = 2\sin(\theta)\cos(\theta)\,d\theta$

As $x \rightarrow 0$ then $\theta \rightarrow 0$, As $x \rightarrow 1$ then $\theta \rightarrow \frac{\pi}{2}$

$$\int_0^{\infty} x^{u-1}\,(1-x)^{v-1}dx$$

$$= \int_0^{\frac{\pi}{2}} [sin^2(\theta)]^{u-1}[cos^2(\theta)]^{v-1}[2\sin(\theta)\cos(\theta)]\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^{2u}(\theta)sin^{-2}(\theta)\,cos^{2v}(\theta)cos^{-2}(\theta)[2\sin(\theta)\cos(\theta)]\,d\theta$$

$$\rightarrow \int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)\,cos^{2v-1}(\theta)d\theta = \frac{\beta(u,v)}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)\,cos^{2v-1}(\theta)d\theta = \frac{\Gamma(u)\Gamma(v)}{2\Gamma(u+v)}$$

Ex$_{11}$/ Evaluate $\int_0^{\frac{\pi}{2}} sin^6(\theta)d\theta$

Sol:

$$\int_0^{\frac{\pi}{2}} sin^6(\theta)d\theta = \int_0^{\frac{\pi}{2}} sin^6(\theta)cos^0(\theta)d\theta$$

Let $2u - 1 = 6 \rightarrow u = \frac{7}{2}$

And $2v - 1 = 0 \rightarrow v = \frac{1}{2}$

$$\rightarrow \int_0^{\frac{\pi}{2}} sin^6(\theta)d\theta = \int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)cos^{2v-1}(\theta)d\theta$$

$$= \int_0^{\frac{\pi}{2}} sin^{2u-1}(\theta)\cos^{2v-1}(\theta)d\theta = \frac{\Gamma(u)\Gamma(v)}{2\Gamma(u+v)}$$

$$\rightarrow \int_0^{\frac{\pi}{2}} sin^{6}(\theta)d\theta = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)} = \frac{5\pi}{32}$$

❖ **Bessel's Equation and Bessel's Function**

The Bessel equation is

$$\bar{\bar{y}} + \frac{1}{x}\bar{y} + \left(1 - \frac{v^2}{x^2}\right)y = 0 \qquad \dots\dots\dots(3)$$

Or, multiply by (x^2)

$$x^2\bar{\bar{y}} + x\,\bar{y} + (x^2 - v^2)y = 0 \qquad \dots\dots\dots(4)$$

Where the parameter (v) is a given real number which is positive or zero and represents the order of the Bessel's equation order.

It is one of the important equations of applied mathematics and engineering mathematics because it is related to the Laplace operator in cylindrical coordinates. The Bessel equation is solved by series solution methods, in fact, to solve the Bessel equation you need to use the method of Frobenius. It might be expected that this method is needed because of the singularities at $(x = 0)$, however, let us pretend we had not noticed and try to use the ordinary series solution method:

First: write the equation of (y) and its

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda} \quad \rightarrow \quad \bar{y} = \sum_{n=0}^{\infty}(n+\lambda)a_n x^{n+\lambda-1}$$

$$\rightarrow$$

$$\bar{\bar{y}} = \sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n x^{n+\lambda-2}$$

Second: Substitute into Bessel’s equation (4), we obtain

$$\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n x^{n+\lambda}+\sum_{n=0}^{\infty}(n+\lambda)a_n x^{n+\lambda}+\sum_{n=0}^{\infty}a_n x^{n+\lambda+2}$$

$$-v^2\sum_{n=0}^{\infty}a_n x^{n+\lambda}=0 \qquad \ldots\ldots\ldots(5)$$

Equate the sum of the coefficients of $(x^{\lambda+r})$ to zero. Note that this power Corresponds to in the first, second, and fourth series, and to $(n=r-2)$ in the third Series. Hence for $(r=0)$ and $(r=1)$, the third series does not contribute since $(n\geq 0)$.

For $r=2,3,3,4,\ldots\ldots$ all four series contribute, so that we get a general formula for all these:

$\lambda(\lambda-1)a_0+\lambda a_0-v^2a_0=0 \qquad \ldots\ldots\ldots(6.a) \qquad for\ r=0$

$\lambda(\lambda+1)a_1+(\lambda+1)a_1-v^2a_1=0 \quad \ldots\ldots\ldots(6.b) \qquad for\ r=1$

$(r+\lambda)(r+\lambda-1)a_r+(r+\lambda)a_r+a_{r-2}-v^2a_r$

$=0 \quad \ldots\ldots\ldots(6.c) \qquad for\ r=2,3,\ldots$

From $(6a)$ we obtain the indicial equation by dropping a_0

$\lambda(\lambda-1)a_0+\lambda a_0-v^2a_0=0 \quad \ldots\ldots\ldots(7)$

$a_0[\lambda^2-\lambda+\lambda-v^2]=0$

$\rightarrow$

$\lambda^2-v^2=0$

The roots are $\lambda_1=v$ & $\lambda_2=-v$

$\therefore \lambda_1-\lambda_2=2v$

This means that the solution depends on the value of (v).

Coefficient recursion:

For $\lambda=\lambda_1=v$.Eq. $(6.b)$ reduces to $[(2v+1)a_1=0]$ Hence $(a_1=0)$ since $v\geq 0$.

Equate the sum of the coefficients of $(x^{\lambda+r})$ to zero:

$(r+\lambda)(r+\lambda-1)a_r x^{r+\lambda} + (r+\lambda)a_r x^{r+\lambda} + a_{r-2}x^{r+\lambda} - v^2 a_r x^{r+\lambda} = 0$

$[(r+\lambda)(r+\lambda-1) + (r+\lambda) - v^2]a_r + a_{r-2} = 0$

$[(r+\lambda)(r+\lambda) - v^2]a_r = -a_{r-2}$

$$a_r = \frac{-a_{r-2}}{(r+\lambda)^2 - v^2} \quad \dots\dots\dots(8)$$

Substituting $\lambda = v$ in (5) gives simply

$$a_r = \frac{-a_{r-2}}{(r+v)^2 - v^2}$$

$$= \frac{-a_{r-2}}{r^2 + 2rv + v^2 - v^2}$$

$$= \frac{-a_{r-2}}{r^2 + 2rv}$$

$= \frac{-a_{r-2}}{r(r+2v)}$ $\quad ,r \geq 2$. And since $a_1 = 0$ then all odd terms will be equal to zero. Let $r = 2m$ for even parts

$\rightarrow$

$$a_{2m} = \frac{-a_{2m-2}}{2^2 m(m+v)} \quad \dots\dots\dots(9)\ for\ m = 1,2,3,\dots.$$

From eq. (9), it can be obtained that:

$$a_2 = \frac{-a_0}{2^2(v+1)}$$

$$a_4 = \frac{-a_2}{2^2 2(v+2)} = \frac{a_0}{2^2 2!\,(v+1)(v+2)}$$

In general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!\,(v+1)(v+2)\dots(v+m)} \quad \dots\dots\dots(10)$$

Now (a_{2m}) is the coefficient of x^{v+2m} in the eq. (3). Hence it would be probably be convenient if (a_{2m}) contained the factor(2^{v+2m}) in the denominator instead of just (2^{2m}) to achieve this, (a_{2m}) is multiplied by (2^v) and getting:

$$a_{2m} = \frac{(-1)^m\,(2^v a_0)}{2^{v+2m} m!(v+1)(v+2)\ldots(v+m)}$$

Since $(v! = \Gamma(v+1))$ then (a_{2m}) is multiplied by (2^v) and getting:

$$a_{2m} = \frac{(-1)^m}{2^{v+2m} m!(v+m)\ldots.(v+2)\ldots\Gamma(v+1)}\,[\,2^v\Gamma(v+1)a_0]$$

Since the gamma function satisfies the recurrence relation $[(v+j)\,\Gamma(v+j) = \Gamma(v+j+1)]$, the factors $(v+1)(v+2)\ldots(v+m)$ can be telescoped into the gamma function, and the expression of a_{2m}can be written as

$$a_{2m} = \frac{(-1)^m}{2^{v+2m} m!\Gamma(v+m+1)}\,[\,2^v\Gamma(v+1)a_0]$$

Since a_0 is still arbitrary, it is convenient to choose:

$a_0 = \frac{1}{2^v\Gamma(v+1)}$ So that

$$a_{2m} = \frac{(-1)^m}{2^{v+2m} m!\Gamma(v+m+1)} \quad \text{For } m = 0,1,2,\ldots\ldots\ldots..$$

The Bessel function of first kind of order (v) is ($J_v(x)$)

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{v+2m}}{2^{v+2m} m!\,\Gamma(v+m+1)} \quad \ldots\ldots\ldots(11)$$

Since $\Gamma(n) = \Gamma(n-1)$ or $\Gamma(n+m+1) = (n+m)!$, for $v = n$ then:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{n+2m}}{2^{n+2m} m!\,(n+m)!} \quad \ldots\ldots\ldots(12)$$

When (v) is not integer and negative then eq. (11) can be written as:

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{-v+2m}}{2^{v+2m} m!\,\Gamma(-v+m+1)} \quad \ldots\ldots\ldots(13)$$

Eq. (13) represents the second particular solution of the Bessel's equation of order (v). This means that the general solution of Bessel's equation is:

$y(t) = c_1 J_v x + c_2 J_{-v}(x)$ ……… (14)

When ($2v$) is integer then $y_2 = B\, y_v(x)$

$\rightarrow y(x) = AJ_v(x) + B\, y_v(x)$ ……… (15)

Now if ($v = 0$) then

$y_2 = B\, y_0(x)$

$\rightarrow y(x) = AJ_0(x) + B\, y_0(x)$ ……… (16)

- ❖ **<u>Derivatives, Recursions</u>**

The derivative of $J_v(x)$ with respect to (x) can be expressed by $J_{v-1}(x)$ or $J_{v+1}(x)$ by the formulas:

$[x^v J_v(x)]' = x^v J_{v-1}(x)$ ……… (17)

$[x^{-v} J_v(x)]' = -x^{-v} J_{v+1}(x)$ ……… (18)

$J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$ ……… (19)

$J_{v-1}(x) - J_{v+1}(x) = 2\bar{J}_v(x)$ ……… (20)

From equations (17&18), it can be found that:

$\int x^v J_{v-1}(x)\, dx = x^v J_v(x) + C$ ……… (21)

$\int x^{-v} J_{v+1}(x)\, dx = -x^v J_v(x) + C$ ……… (22)

$\bar{J}_v(x) = -\frac{k}{x} J_v(x) + J_{v-1}(x)$ ……… (23)

$\bar{J}_v(x) = \frac{k}{x} J_v(x) - J_{v+1}(x)$ ……… (24)

Ex$_{13}$/ prove that $[x^v J_v(x)]' = x^v J_{v-1}(x)$

Sol:

The right side

$$\frac{d}{dx}\,x^{v}\left[\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{v+2m}}{2^{v+2m}m!\,\Gamma(v+m+1)}\right]$$

$$=\frac{d}{dx}\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{2v+2m}}{2^{v+2m}m!\,\Gamma(v+m+1)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m}2(v+m)x^{2v+2m-1}}{2^{v+2m}m!\Gamma(v+m+1)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m}(v+m)x^{2v+2m-1}}{2^{v+2m-1}m!(v+m)\Gamma(v+m)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{2v+2m-1}}{2^{v+2m-1}m!\Gamma(v+m)}$$

The left side

$$x^{v}J_{v-1}(x)=x^{v}\left[\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{v+2m-1}}{2^{v+2m-1}m!\Gamma(v+m+1-1)}\right]$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m}x^{2v+2m-1}}{2^{v+2m-1}m!\Gamma(v+m)}$$

∴The left side = the right side

Ex14/ prove that $\frac{d}{dx}[xJ_v(x)J_{v+1}(x)]=x[J^2{}_v(x)-J^2{}_{v+1}(x)]$

The right side

$$\frac{d}{dx}[xJ_v(x)J_{v+1}(x)]=xJ_v(x)\bar{J}_{v+1}(x)+J_{v+1}(x)[x\bar{J}_v(x)+J_v(x)]$$

$$=\quad xJ_v(x)\left[-\frac{v+1}{x}J_{v+1}(x)+J_v(x)\right]+J_{v+1}(x)[x\left(\frac{v}{x}J_v(x)-J_{v+1}(x)\right)+J_v(x)]$$

$$=\frac{-xvJ_v(x)J_{v+1}(x)}{x}-\frac{xJ_v(x)J_{v+1}(x)}{x}+xJ^2{}_v(x)+\ vJ_{v+1}(x)\,J_v(x)-xJ^2{}_{v+1}(x)+J_v(x)J_{v+1}(x)$$

$$=-vJ_v(x)J_{v+1}(x)-J_v(x)J_{v+1}(x)+vJ_v(x)J_{v+1}(x)$$

$+J_v(x)J_{v+1}(x)+xJ^2{}_v(x)-xJ^2{}_{v+1}(x)$

$=x[J^2{}_v(x)-J^2{}_{v+1}(x)]$

Ex15/ evaluate $\int x^4 J_3(2x)dx$

Sol:

Let $y=2x\ \rightarrow x=\frac{y}{2}$ & $dx=\frac{dy}{2}$

$\rightarrow \int x^4 J_3(2x)dx=\int \frac{y}{2^5}^4 J_3(y)dy$

$=\frac{y}{2^5}^4 J_4(y)+c$

$=\frac{x^4}{2}J_4(y)+c$

Ex16/find the result of the following integral $\int xln(x)\,J_0(x)dx$

Sol:

Let $u=\ln(x)\rightarrow du=\frac{1}{x}$ & $dv=xJ_0(x)dx\ \rightarrow v=xJ_1(x)$

$\rightarrow \int xln(x)J_0(x)dx=\ln(x)\,xJ_1(x)\ -\int J_1(x)dx$

But $\int J_1(x)dx\ =\ -x^{-0}J_0(x)$ then

$\int xln(x)J_0(x)dx=\ln(x)\,xJ_1(x)\ +J_0(x)+c$

Ex17/ evaluate $J_3(x)$

Sol:

Since $J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x)$ then

$J_{v+1}(x)=\frac{2v}{x}J_v(x)-J_{v-1}(x)$

$\rightarrow J_3(x)=\frac{4}{x}J_2(x)-J_1(x)$

And since $J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$

$\rightarrow J_3(x)=\frac{4}{x}\left[\frac{2}{x}J_1(x)-J_0(x)\right]-J_1(x)$

$$\therefore J_3(x) = \frac{8}{x} J_1(x) - J_1(x) - \frac{4}{x} J_0(x)$$

Remark[1]: when $v = \frac{1}{2}$ then $J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m} x^{\sqrt{2}}}{2^{\frac{1}{2}+2m} m! \Gamma\left(m+\frac{3}{2}\right)}$ ……… (26)

Ex[18]/ find $J_{\frac{1}{2}}(x)$

Sol:

$$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m} x^{\sqrt{2}}}{2^{\frac{1}{2}+2m} m! \, \Gamma\left(m+\frac{3}{2}\right)}$$

$J_{\frac{1}{2}}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m} x^{\sqrt{2}}}{2^{\frac{1}{2}+2m} m! \Gamma\left(m+1+\frac{1}{2}\right)}$, but $\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}$

Then $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$

$\rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x)$

In the same way

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x)$$

Ex[19]/ find the result of $J_{\frac{5}{2}}(x)$

Sol:

Since $J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$then $J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x)$

$\rightarrow J_{\frac{5}{2}}(x) = \frac{2\frac{3}{2}}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$

$$J_{\frac{3}{2}}(x) = \frac{2\frac{1}{2}}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\rightarrow J_{\frac{5}{2}}(x) = \frac{3}{x}\left[\frac{1}{x} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)\right] - J_{\frac{1}{2}}(x)$$

$$= \frac{3}{x^2} J_{\frac{1}{2}}(x) - \frac{3}{x} J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= \left(\frac{3}{x^2} - 1\right) \sqrt{\frac{2}{\pi x}} \sin(x) - \frac{3}{x} \sqrt{\frac{2}{\pi x}} \cos(x)$$

❖ <u>Modified Bessel's Equation</u>

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind. In addition, are defined by any of these equivalent alternatives:

$$I_{\propto}(x) = i^{-\propto} J_{\propto}(ix) = \sum_{m=0}^{\infty} \frac{(\frac{x}{2})^{2m+\propto}}{m!\,\Gamma(m+\propto+1)} \quad \ldots\ldots\ldots (27)$$

$$k_{\propto}(x) = \frac{\pi}{2} \frac{I_{-\propto}(x) - I_{\propto}(x)}{\sin(\propto \pi)} = \frac{\pi}{2} i^{\propto+1} H_{\propto}{}^{1}(ix) \quad \ldots\ldots\ldots (28)$$

These are chosen to be real-valued for real and positive arguments x. The series expansion for $I_{\propto}(x)$ is thus similar to that for $J_{\propto}(x)$ but without the alternating $(-1)^m$ factor. $I_{\propto}(x)$ & $k_{\propto}(x)$ are the two linearly independent solutions to the modified Bessel's equation:

$$x^2 \bar{\bar{y}} + x\bar{y} - (x^2 + \propto^2)y = 0 \ldots\ldots\ldots (29)$$