Z-Transform
Part one

❖ Introduction

In the study of discrete-time signal and systems, we have thus far considered the time-domain and the frequency domain. The z-domain gives us a third representation. All three domains are related to each other. Recalling to the Laplace Transform (L.T), it can be found that L.T is used for continuous time signals as illustrated in the L.T. formula

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad \ldots\ldots\ldots (1)$$

For discrete time functions, the summation is used to be equivalent to the integration of continuous functions as given in equation (2)

$$Z\,[f(n)] = \sum_{n=0}^{\infty} f(n)\,e^{-sn} \quad \ldots\ldots\ldots (2)$$

Assume that Z = e^s then eq. (2) for one side Z- transform can be written as

$$Z\,[f(n)] = \sum_{n=0}^{\infty} f(n)\,Z^{-n} \quad \ldots\ldots\ldots (3)$$

In this transformation, two domains are present so that

$f(n) \rightarrow Z\,[f(n)] \rightarrow f(z)\ \&\ f(z) \rightarrow Z\,[f(n)]^{-1} \rightarrow f(n)$

Where Z is a complex variable where $Z = re^{jw}$

❖ Range of Convergence(RoC)

As in the case of the Laplace transform, the range of values of the complex variable Z for which the z-transform converges is called the region of convergence. The (RoC) has the following properties:

1- The RoC does not contain any poles.

2- If $x[n]$ is a finite sequence (that is, $x[n] = 0$ except in a finite interval $N_l \leq n \leq N_2$ where N_l, and N_2, are finite) and $X(z)$

converges for some value of z, then the RoC is the entire z-plane except possibly $z = 0$ *or* $z = \infty$.

3- If $x[n]$ is a right-sided sequence (that is, $x[n] = 0$ for $n < N_1 < \infty$) and $X(z)$ converges for some value of z, then the RoC is of the form

$$|Z| > r_{max} \quad \& \quad \infty > |Z| > r_{max}$$

Where r_{max} , equals the largest magnitude of any of the poles of $X(z)$. Thus, the RoC is the exterior of the circle $|Z| = r_{max}$, in the z-plane with the possible exception of $z = m$.

4- If $x[n]$ is a left-sided sequence (that is, $x[n] = 0$ for $n > N_2 > -\infty$) and $X(z)$ converges for some value of z, then the RoC is of the form

$$|Z| < r_{min} \quad \& \quad 0 < |Z| < r_{min}$$

Where ($r_{\min}$), is the smallest magnitude of any of the poles of $X(z)$. Thus, the RoC is the interior of the circle $|Z| = r_{min}$ in the z-plane with the possible exception of $z = 0$.

5- If $x[n]$ is a two-sided sequence (that is, $x[n]$ is an infinite-duration sequence that is neither right-sided nor left-sided) and $X(z)$ converges for some value of z, then the RoC is of the form

$$r_1 < |Z| < r_2$$

Where (r_1 & r_2) are the magnitudes of the two poles of $X(z)$. Thus, the RoC is an annular ring in the z-plane between the circles $|Z| = r_2$, and $|Z| = r_2$ not containing any poles.

Note that Property one follows immediately from the definition of poles; that is, $X(z)$ is infinite at a pole.

Ex₁/ find Z- transform for unit impulse function

Sol:

The discrete unit impulse (Dirac delta) function is described by the following equation:

$$\delta(n) = \begin{bmatrix} 1 & n = 0 \\ 0 & n \neq 0 \end{bmatrix} \quad \dots\dots\dots(4)$$

$$\delta(z) = \sum_{n=0}^{\infty} \delta(n)\, Z^{-n}$$

$= 1 + 0 * Z^{-1} + 0 * Z^{-2} + \dots\dots$ etc.

$\therefore Z[\delta(n)] = 1$

Ex₂/ what is the Z-transform of the unit step function

Sol:

The unit step function is expressed in equation (5)

$$U(n) = \begin{bmatrix} 1 & n \geq 0 \\ 0 & n < 0 \end{bmatrix} \quad \dots\dots\dots(5)$$

$Z[U(n)] = \sum_{n=0}^{\infty} U(n)\, Z^{-n}$

$\rightarrow S_n = 1 + Z^{-1} + Z^{-2} + Z^{-3} + Z^{-4} + \dots\dots.. + Z^{-n} \quad \dots\dots\dots(6)$

Multiply by Z^{-1}

$Z^{-1}S_n = Z^{-1} + Z^{-2} + Z^{-3} + Z^{-4} + \dots\dots.. + Z^{-n} + Z^{-n-1} \quad \dots\dots\dots(7)$

Subtract eq. (7) from eq. (6)

$S_n - Z^{-1}S_n = 1 - Z^{-n-1}$

$\rightarrow S_n(1 - Z^{-1}) = 1 - Z^{-n-1}$

$\rightarrow S_n = \frac{1-Z^{-n-1}}{1-Z^{-1}}$ .To test convergence then $\lim_{n\to\infty}\left(\frac{1}{1-z^{-1}} - \frac{z^{-(n+1)}}{1-z^{-1}}\right)$

$= \frac{1}{1-z^{-1}} + 0$

$$\rightarrow Z[U(n)] = \frac{1}{1+\frac{1}{z}} \quad \therefore \; Z[U(n)] = \frac{z}{1+z}$$

Ex3/ find Z- transform for the ramp function

Sol:

The ramp function is $f(n) = n$

$$\rightarrow Z[f(n)] = \sum_{n=0}^{\infty} nZ^{-n}$$

$S_n = 0 + Z^{-1} + 2Z^{-2} + 3Z^{-3} + \cdots \ldots\ldots\ldots + nZ^{-n} \ldots\ldots\ldots (8)$

Multiply by Z^{-1}

$Z^{-1}S_n = Z^{-2} + 2Z^{-3} + 3Z^{-4} + \cdots \ldots\ldots\ldots + nZ^{-(n+1)} \ldots\ldots\ldots (9)$

Subtract eq. (9) from eq. (8)

$(1-Z^{-1})S_n = Z^{-1} + Z^{-2} + Z^{-3} + \cdots - nZ^{-(n+1)} \ldots\ldots\ldots (10)$

Again, multiply by Z^{-1}

$(1-Z^{-1})Z^{-1}S_n = Z^{-2} + Z^{-3} + Z^{-4} + \cdots - nZ^{-(n+2)} \ldots\ldots\ldots (11)$

Subtract eq. (11) from eq. (10)

$[(1-Z^{-1}) - (1 - Z^{-1})Z^{-1}]S_n = Z^{-1} + nZ^{-(n+2)}$

$\rightarrow S_n = \frac{Z^{-1}+nZ^{-(n+2)}}{(1-Z^{-1})^2}$ .To find the convergence then

$$\lim_{n\to\infty} \frac{Z^{-1}}{(1-Z^{-1})^2} + \lim_{n\to\infty} \frac{nZ^{-(n+2)}}{(1-Z^{-1})^2}$$

$$= \frac{Z^{-1}}{(1-Z^{-1})^2} + 0$$

$$\rightarrow Z[n] = \frac{Z^{-1}}{(1-Z^{-1})^2}$$

$$= \frac{Z^{-1}}{(1-\frac{1}{z})^2}$$

$$= \frac{Z^{-1}}{\frac{(z-1)^2}{z^2}}$$

$$\therefore Z[n] = \frac{z}{(z-1)^2}$$

Z-Transform

Part one

Ex$_4$/ prove that $Z[a^n] = \frac{z}{z-a}$

Sol:

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-1} = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ \dots\dots.$$

$a = 1, r = az^{-1}$, which converge to $\frac{a}{a-r} \rightarrow \ Z[a^n] = \frac{1}{1-az^{-1}}$

$$\therefore \ Z[a^n] = \frac{z}{z-a}$$

Ex$_5$/ find Z-transform of $\cos(bn)$

Sol:

$$Z[\cos(bn)] = Z[\frac{e^{Jbn}+e^{-Jbn}}{2}]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{Jbn}} + \frac{z}{z-e^{-Jbn}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-Jbn}+z-e^{Jbn}}{z^2-ze^{Jbn}-ze^{-Jbn}+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{Jbn}+e^{-Jbn})}{z^2-z[e^{Jbn}+ze^{-Jbn}]+1}\right]$$

$$\therefore Z[\cos(bn)] = \frac{z(z-cosb)}{z^2-2zcosb+1}$$

HW$_1$/ find $Z[\sin(bn)]$

HW$_2$/ prove that Z $[e^{-an}\sin(bn)] = \frac{ze^{-a}\sin(bn)}{z^2-ze^{-a}\cos(b)+e^{-2a}}$

Ex$_6$/ find Z –transform for the function that shown in Figure (1)

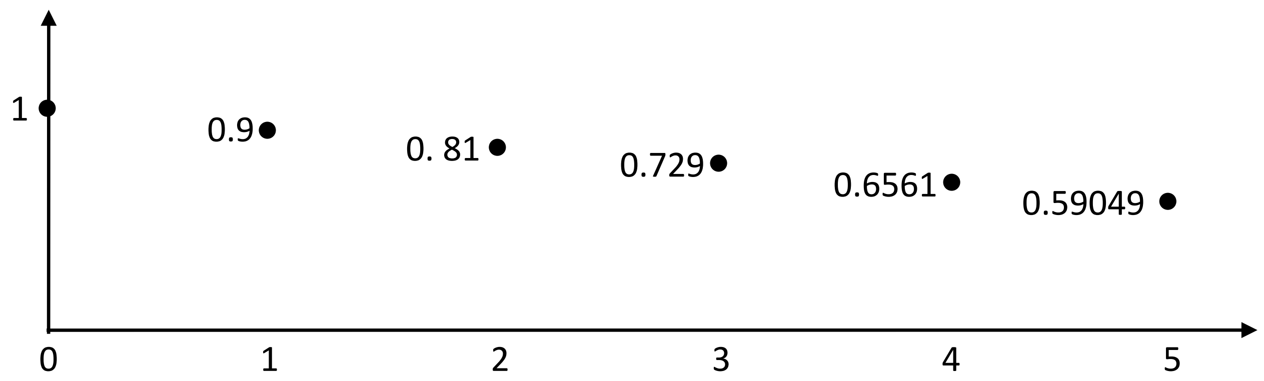

Fig (1)

Sol:

Since $Z[f(n)] = \sum_{n=0}^{\infty} f(n)Z^{-n}$

$=f(0) + f(1)Z^{-1} + f(2)Z^{-2} + f(3)Z^{-3} + f(4)Z^{-4} + \cdots$

From Fig (1), it can be found that

$f(0) = 1\ , f(1) = 0.9, f(2) = 0.81 = (0.9)^2, f(3) = 0.729 = (0.9)^3$

$f(4) = 0.6561 = (0.9)^4, f(5) = 0.59049 = (0.9)^5, \ldots$

$\rightarrow$

$Z[f(n)] = 1 + 0.9Z^{-1} + (0.9Z^{-1})^2 + (0.9Z^{-1})^3 + (0.9Z^{-1})^4 + \cdots$

$a = 1\ \&\ r = 0.9Z^{-1}$According to the power series condition, it can be obtained that:

$$Z[f(n)] = \frac{1}{1 - 0.9z^{-1}}$$

$$= \frac{1}{1 - \frac{0.9}{z}} \rightarrow Z[f(n)] = \frac{z}{z - 0.9}$$

Ex7/ sketch the following $F(z) = \frac{2z}{z+0.5}$ for five terms

Sol:

$$2 - Z^{-1} + 0.5Z^{-2} - 0.25Z^{-3} + 0.125Z^{-4} - 0.0625Z^{-5}$$

$z + 0.5 \,\big)\, 2z$

$\mp 2z \mp 1$

-1

$\pm 1 \pm 0.5z^{-1}$

$0.5z^{-1}$

$\mp 0.5z^{-1} \mp 0.25z^{-2}$

$-0.25z^{-2}$

$\pm 0.25z^{-2} \pm 0.125z^{-3}$

$0.125z^{-3}$

$\mp 0.125z^{-3} \mp 0.0625z^{-4}$

$-00.0625z^{-4}$

$\pm 00.0625z^{-4} \pm 0.0312525z^{-5}$

$0.0312525z^{-5}$

This means that $F(z) = 2 - Z^{-1} + 0.5Z^{-2} - 0.25Z^{-3} + 0.125Z^{-4} - 0.0625Z^{-5} + \ ...$

Now this function can be drawn as given in Fig (2).

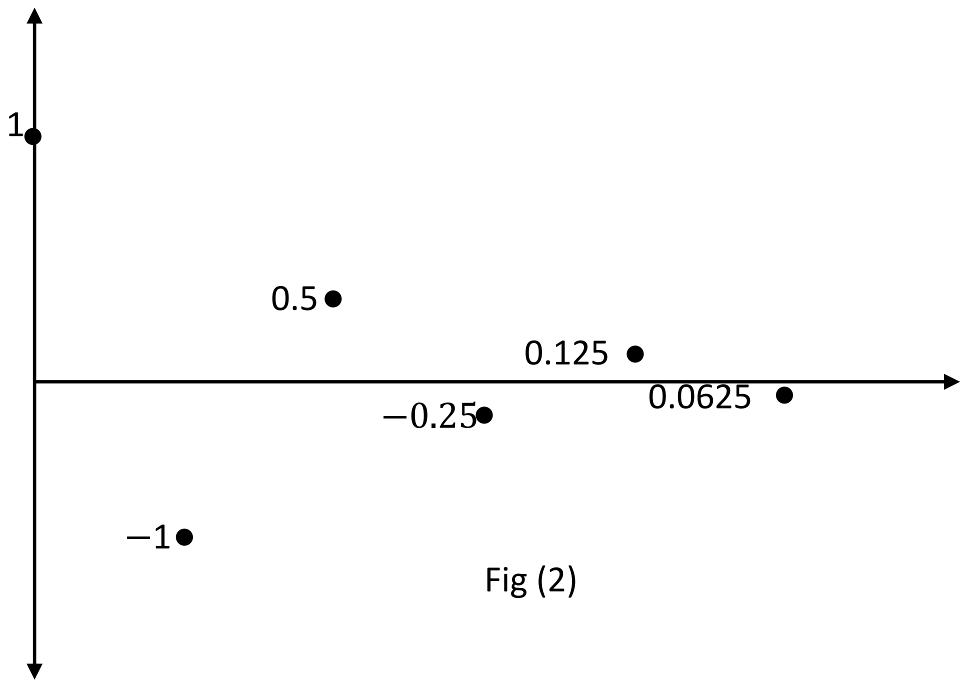

Fig (2)

HW3/ sketch the following $F(z) = \frac{z}{z+2}$ for four terms

❖ **Properties of Z−Transform**

1- Linearity

$Z[c_1 f_1(n)] + Z[c_2 f_2(n)] + \ ... = \ c_1 Z[f_1(n)] + c_2 Z[f_2(n)] + \cdots$

Where c_1 & c_2 are constants

2- Right shifting property (delay in time)

$Z[f(n - n_o)u(n - n_0)] = z^{-n_o} F(z)$ Note that $n_0 \geq 0$

$$f(n - n_o)u(n - n_o) = \begin{cases} 0 & n < n_0 \\ f(n - n_0) & n \geq n_0 \end{cases}$$

3- Left shifting property (advance in time)

$$Z[f(n + n_o)u(n)] = z^{no}[F(z) - \sum_{n=0}^{no-1} f(n) z^{-n}]$$

4- Multiplying by (e^{-an})

$Z[f(n)e^{-an}] = F(ze^{a})$

5- Initial value property

This property is considered to find $f(0)$ which can be obtained directly by put $(n = 0)$ in the function $f(n)$ or by taking $\lim_{z\to\infty} F(z)$ and choose the first term that not contains (z^{-1}).

6- Final value property

This value represent $f(\infty)$ or can be found from the following equation

$$\lim_{z\to 1}(z-1)F(z)$$

7- Multiplication by (n) or (Differentiation in Z)

$$Z[n\,f(n)] = -z\,\frac{d}{dz}\,F(z)$$

8- Accumulation

$$\sum_{k=-\infty}^{n} x(k) = \frac{1}{1-z^{-1}}x(z) = \frac{z}{z-1}x(z)$$

9- Convolution

$x_1(\mathrm{n}) * x_2(\mathrm{n}) \leftrightarrow \mathrm{X}_1(\mathrm{Z}).\mathrm{X}_2(\mathrm{Z})$

This relationship plays a central role in the analysis and design of discrete-time systems.

10-Time scaling property

$Z[f\left(\frac{n}{k}\right) = f(z^{k})$. Where (k) integer and $(k \neq 0)$

Ex$_8$/ find $Z[\sin(n-a)\,u(n-n_0)]$

Sol:

Since$Z[f(n-n_o)u(n-n_0)] = z^{-n_o}F(z), Z[\sin(bn)] = \frac{z\sin(b)}{z^2-2z\cos b+1}$,

and b=1, then $Z[\sin(n-a)\,u(n-n_0)] = z^{-1}\,\frac{z\sin(1)}{z^2-2z\cos(1)+1}$

Ex9/ find Z $[e^{-cn}\cos(60n)]$

Sol:

Since $Z[f(n)e^{-an}] = F(ze^{a})$, and $Z[\cos(bn)] = \frac{z(z-cosb)}{z^2-2zcosb+1}$

$\therefore$ Z $[e^{-cn}\cos(30n)] = \frac{ze^{c}(ze^{c}-cos60)}{z^2e^{2c}-2ze^{c}cos60+1}$

$= \frac{ze^{c}(ze^{c}-\frac{1}{0.5})}{z^2e^{2c}-2ze^{c}\frac{1}{0.5}+1}$

Ex10/ find $Z[e^{(n+3)}]$ in two ways

Sol:

First:

$e^{(n+3)} = e^{n}e^{3} \rightarrow Z[e^{(n+3)}] = Z[e^{n}e^{3}]$

$= e^{3}Z[e^{n}]$.since $Z[a^{n}] = \frac{z}{z-a}$

$\rightarrow Z[e^{n)}] = \frac{z}{z-e^{1}} \therefore e^{3}Z[e^{n}] = e^{3}\frac{z}{z-e^{1}}$

$$= \frac{e^{3}z}{z-e^{1}}$$

Second:

$$Z[e^{n)}] = \frac{z}{z-e^{1}} \therefore Z[e^{(n+3)}] = z^{3}[\frac{z}{z-e^{1}} - \sum_{n=0}^{2} e^{n}z^{-n}]$$

$$= \frac{z^{4}}{z-e^{1}} - z^{3}[e^{0}z^{0} + e^{1}z^{-1} + e^{2}z^{-2}]$$

$$= \frac{z^{4}}{z-e^{1}} - z^{3} - e^{1}z^{2} - e^{2}z^{1}$$

$= \frac{z^4-z^4+z^{3e^1}-z^3e^1+z^2e^2-z^2e^2+e^3}{z-e^1}$

$= \frac{e^{3}z}{z-e^{1}}$

Ex$_{10}$/ apply initial value theorem for $cos(n)$

Sol:

$f(0) = \lim_{z\to\infty} F(z)$

$$cos(0) = \lim_{z\to\infty} \frac{z(z-cos1)}{z^2-2zcos1+1} \rightarrow cos(0) = \lim_{z\to\infty} \frac{z^2-cos1}{z^2-2zcos1+1}$$

Dividing on $(z^2) \rightarrow cos(0) = \lim_{z\to\infty} \frac{1-\frac{cos1}{z^2}}{1-\frac{2}{z}cos1+\frac{1}{z^2}}$

$$cos(0) = \frac{1-0}{1-0+0} \rightarrow 1 = 1$$

Ex$_{11}$/ find $Z[n3^n]$

Sol:

$$Z(3^n) = \frac{z}{z-3} \rightarrow \bar{F}(z) = \frac{-3}{(z-3)^2}$$

$$Z[n3^n] = \frac{-z*-3}{(z-3)^2}$$

Ex$_{12}$/ find $Z\left[\frac{n}{4}\right]$

Sol:

Since$Z[n] = \frac{z}{(z+1)^2}$, then $Z\left[\frac{n}{4}\right] = \frac{z^4}{(z^4+1)^2}$

- **<u>Inverse of Z-transform</u>**

There are three methods to find z-transform, which are:

1- Direct method

By comparison with z-transform of mathematics functions and apply the properties of z-transform, the function can be found as illustrated in the following example:

Ex$_{13}$/ what is $f(n)$for $\frac{z(z-0.25)}{z^2-0.5z+0.25}$

Sol: since $Z[cosbn] = \frac{z(z-cosb)}{z^2-2zcosb+1}$, and $Z[f(n)e^{-an}] = F(ze^a)$, then

$$Z[cosbn] = \frac{e^a z(z - cosb)}{z^2 e^{2a} - 2ze^a cosb + 1}$$

$= \frac{z(z-e^{-a}cosb)}{z^2-2ze^{-a}cosb+e^{-2a}}$

$\therefore e^{-2a} = 0.25 \rightarrow e^{-a} = 0.5$

$\ln(e^{-2a}) = \ln(0.25)$

$-2a = \ln(0.25) \rightarrow a = \frac{-\ln(0.25)}{2} \cong 0.7$

And $2e^{-a}cosb = 0.5 \rightarrow cosb = 0.5$

$b = cos^{-1}(0.5) \rightarrow b = \frac{\pi}{3}$

This means that $f(n) = e^{-0.7}cos\frac{\pi}{3}n$

2- Partial-Fraction Expansion

As in, the case of the inverse Laplace transforms the partial-fraction expansion method provides the most generally useful inverse z-transform, especially when $X(z)$ is a rational function of(z). Let

$$X(z) = \frac{N(z)}{M(z)} = k\frac{(z - z_1)(z - z_2) \dots (z - z_n)}{(z - z_1)(z - z_2) \dots (z - z_m)}$$

Where (k) is constant.

Ex$_{14}$/ find $Z^{-1}$for the following $(\frac{z+7}{(z-3)(z+2)})$

Sol:

Since $F(z) = (\frac{z+7}{(z-3)(z+2)})$, then

$(\frac{z+7}{(z-3)(z+2)}) = \frac{A}{(z-3)} + \frac{B}{(z+2)}$

$\rightarrow Az + 2A + Bz - 3B = z + 7$

$A + B = 1 \dots\dots\dots(12)$

$2A - 3B = 7 \dots\dots\dots(13)$

Multiplying eq. (12) by (2) and subtract this equation from eq. (13)

$2A + 2B = 2$

$\mp 2A \pm 3B = \mp 7$

$5B = -5 \rightarrow B = -1$ Substituting in eq. (12) $\rightarrow A - 1 = 1$

$\therefore F(z) = \frac{2}{(z-3)} - \frac{1}{(z+2)}$ Comparing with $F(a^n) = \frac{z}{(z-a)}$

& $F(a^{n-n_0})z^{-n_0}F(z)$ then

$$F(z) = \frac{2z^{-1}z}{(z-3)} - \frac{1z^{-1}z}{(z+2)}$$

$\rightarrow f(n) = [2(3)^{n-1} - (-2)^{n-1}]u(n-1)$

3- Complex – Inverse Integral

$$f(n)\frac{1}{2\pi J}\oint Z^{n-1}dz \ldots\ldots\ldots (14)$$

4- Power Series Expansion

The defining expression for the z-transform eq. (3) is a power series where the sequence values $x[n]$ are the coefficients of z^{-n}. Thus, if $f(\,z)$ is given as a power series in the form:

$Z\,[f(n)] = \sum_{n=0}^{\infty} f(n)\,Z^{-n}$

$= f(0)Z^0 + f(1)Z^1 + f(2)Z^2 + \cdots$

Any particular value of the sequence can be determined by finding the coefficient of the appropriate power of Z^{-1} . This approach may not provide a closed-form solution but is very useful for a finite-length sequence where X(z) may have no simpler form than a polynomial in Z^{-1}. For rational r-transforms, a power series expansion can be obtained by long division.

- **<u>THE SYSTEM FUNCTION OF DISCRETE-TIMESYSTEMS</u>**

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values x[n] into an output sequence with values y[n]. This can be denoted as:

$y[n] = T\{x[n]\} \ldots\ldots\ldots (15)$

There some important remarks that must be taken into acount which are:

Remrk₁/ The ideal delay system is defined as:

$y[n] = x[n - n_0] \qquad -\infty < n < \infty$

Where (n_0) is a fixed positive integer representing the delay of the system. In other words, the ideal delay system shifts the input sequence to the right by (n_0) samples to form the output. The types of discrete time system are:

1- Systems without memory (memory less)

A system is said to be memory less if the output for each value of the independent variable at a given time n depends only on the input value at time n. For example system specified by the relationship

$y[n] = \cos(x[n]) + z$

Is memory less, a particularly simple memory less system is the identity system defined by:

$y[n] = x[n]$

In general, the relationship for memory less input-output system can be written as:

$y[n] = g(x[n])$

These systems are called statics systems.

2- Systems with memory

A simple example of system with memory is a delay defined by

$y[n] = x[n - 1]$

A system with memory retains or stores information about input values at times other than the current input value. These systems are called dynamic systems. In these systems, the output is a function of the present and last input values.

For example: $y[n] = 0.5\, x[n] - 2x[n-1] + x[n-2] \dots\dots\dots (16)$

This system can be represented using block diagram as shown in figure (3)

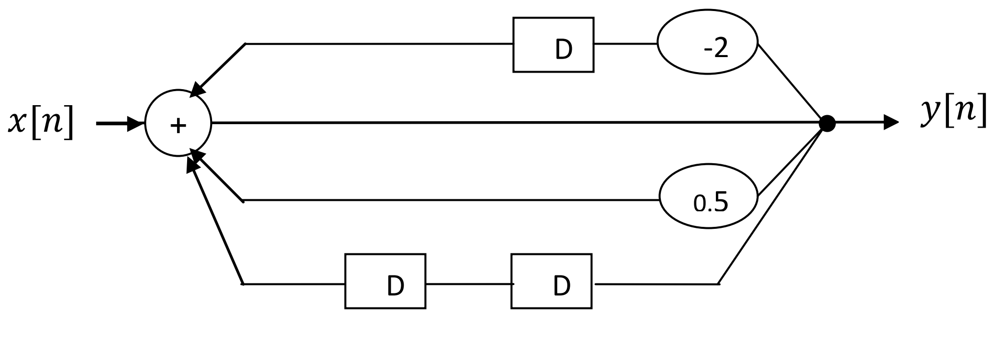

Fig (3)

Ex$_{15}$/ for the block diagram shown in figure (4), find the impulse response by using Z- transform. Note that all initial conditions are zeros.

Sol:

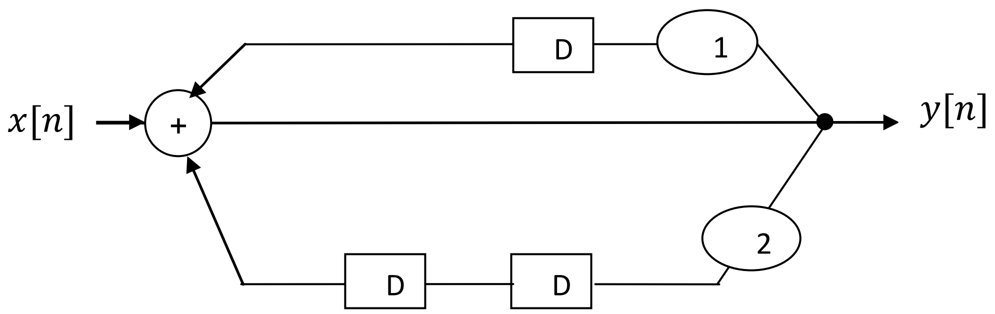

Fig (4)

Sol:

From Fig (4)

$y[n] = x[n] + y[n-1] + 2y[n-2]$

Taking Z-T for two sides

$y[z] = x[z] + z^{-1}y(z) + 2z^{-2}y(z)$

Since the input is unit impulse, then $x[z] = 1 \rightarrow$

$y[z] = 1 + z^{-1}y(z) + 2z^{-2}y(z)$

$\rightarrow y[z] - z^{-1}y(z) - 2z^{-2}y(z) = 1$

$\rightarrow y[z](1 - z^{-1} - 2z^{-2}) = 1$

$$\rightarrow y[z] = \frac{1}{(1 - z^{-1} - 2z^{-2})}$$

$\rightarrow y[z] = \frac{z^2}{(z^2-z^1-2)}$

$\therefore y[z] = \dfrac{z^2}{(z-2)(z+1)}$

Using partial fraction method to find $y[n]$

$\dfrac{y[z]}{z} = \dfrac{z}{(z-2)(z+1)} \rightarrow$

$\dfrac{y[z]}{z} = \dfrac{A}{(z-2)} + \dfrac{B}{(z+1)} \rightarrow$

$Az + A + Bz - 2B = z$

$A + B = 1 \quad \dots\dots\dots (17)$

$A - 2B = 0 \dots\dots\dots (18)$ Subtract eq. (18) from (17) then

$A + B = 1 \quad \dots\dots\dots (17)$

$\mp A \pm 2B = 0 \dots\dots\dots (18)$

$3B = 1 \rightarrow B = \frac{1}{3}$, substituting in eq. (17) then $A + \frac{1}{3} = 1 \therefore\ A = \frac{2}{3}$

Now

$\dfrac{y[z]}{z} = \dfrac{\frac{2}{3}}{(z-2)} + \dfrac{\frac{1}{3}}{(z+1)} \rightarrow$

$y[z] = \dfrac{\frac{2}{3}\,z}{(z-2)} + \dfrac{\frac{1}{3}z}{(z+1)} \rightarrow\ y[n] = \dfrac{2}{3}(2)^n + \dfrac{1}{3}(-1)^n$

Ex$_{16}$/ Solve the following difference equation

$f[n+2]u(n) - 2f[n] + f[n-1]u(n-n_0) = 0$

If $f(0) = 0$ & $f(0) = 1$

Sol:

Since $Z[f(n+n_o)u(n)] = z^{no}[F(z) - \sum_{n=0}^{no-1} f(n)z^{-n}]$ then

$$f[n+2] = z^2[F(z) - \sum_{n=0}^{1} f(n)z^{-n}]$$

And since $Z[f(n-n_o)u(n-n_0)] = z^{-n_o}F(z)$ then

$Z[f(n-1)u(n-1)] = z^{-1}F(z)$

$\rightarrow\ z^2F(z) - z^2f(0) - z^2f(1)z^{-1} - 2f(z) + z^{-1}F(z) = 0$

For initial conditions

$z^2F(z) - z^1 - 2F(z) + z^{-1}F(z) = 0$

$$F(z) = \frac{z}{z^2+z-2}$$

$=\frac{z}{(z+2)(z-1)}$ using partial fraction $\frac{F(z)}{z} = \frac{A}{(z+2)} + \frac{B}{(z-1)}$

$Az - A + Bz + 2B = 1$

$A + B = 0 \quad \ldots\ldots\ldots (19)$

$-A + 2B = 1 \ldots\ldots\ldots (20)$ by adding eq. (18) and (17) then

$3B = 1 \quad \ldots\ldots\ldots (19)$

$3B = 1 \rightarrow B = \frac{1}{3}$, substituting in eq. (19) then $A + \frac{1}{3} = 0 \therefore\ A = \frac{-1}{3}$

Now

$$\frac{F[z]}{z} = \frac{\frac{-1}{3}}{(z-2)} + \frac{\frac{1}{3}}{(z+1)} \rightarrow$$

$$F[z] = \frac{1}{3}\left[\frac{z}{(z-1)} - \frac{z}{(z+2)}\right] \rightarrow\ f[n] = \frac{1}{3}[(1)^n - (-2)^n]$$

Ex17/ for the following Z-T, find the system equation.

$$y[z] = \frac{0.5z}{z^2+5z+6}$$

Sol:

$$y[z] = \frac{0.5z}{z^2+5z+6}$$

$$y[z] = \frac{0.5}{z+5+6z^{-1}}$$

$\rightarrow zy[z] + 6z^{-1}y[z] + 5y[z] = 0.5$

$y[n+1] + 6z^{-1}y[n] + 5y[n] = 0.5x[n]$

3- Linear System

The class of linear systems is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$are the responses of a system when $x_1[n]$and $x_2[n]$ are the respective inputs, then the system is linear if and only if

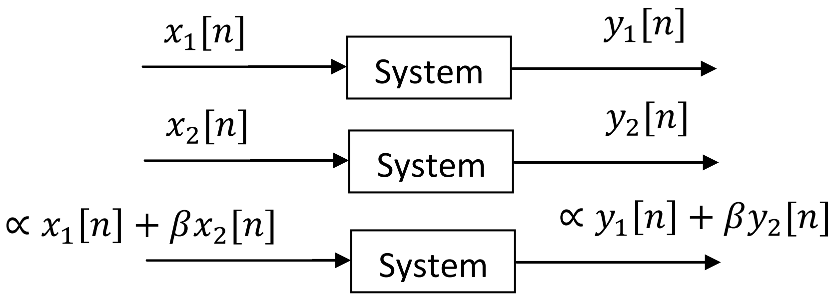

Fig (5)

4- Non Linear System

If the condition that shown in figure (5) is not true for an arbitrary system, then the system is called non linear system, the following two examples will illustrate three and four systems.

Ex$_{18}$/ prove that $y[n] = x[n]$ is linear system.

Sol:

If $y_1[n] = ax_1[n]$, $y_2[n] = bx_2[n]$, and $y_3[n] = x_3[n]$

Now if $x_3[n] = ax_1[n] + bx_2[n]$

$\rightarrow y_3[n] = ax_1[n] + bx_2[n]$

$\rightarrow y_3[n] = y_1[n] + \ y_2[n]$

$\therefore$ This system is linear

Ex$_{18}$/ is $y[n] = log_{10}(x[n])$ linear system.

Sol:

For this system:

$y_1[n] = log_{10}(x_1[n])$, $y_2[n] = log_{10}(x_2[n])$ and $y_3[n] = log_{10}(x_3[n])$

Now if $x_3[n] = ax_1[n] + bx_2[n]$

$\rightarrow y_3[n] = log_{10}(ax_1[n] + bx_2[n])$

This is not equal to $(y_1[n] + y_2[n])$ or $[log_{10}(x_1[n])+ log_{10}(x_2[n])]$

∴This system is non-linear system.

5- Time-Invariant System

The system is called time-invariant system if for input $x_1[n]$,there is an output $y_1[n]$, and when the input is shifted by (n_0) or for $\{x_1[n - n_0]\}$ then the output will be shifted by (n_0) or $\{y_1[n - n_0]\}$.

6- Causal System

A system is causal if the output at anytime depends only on values of the input at the present time and in the past.

$y[n] = f(x[n], x[n-1], \ldots)$

All memoryless systems are causal.

Otherwise, if a system output depends on the future input values, such as $[n + 1], x[n + 2], \ldots\ldots\ldots.$, the system is noncausal. The noncausul system cannot be realized in real time.

Ex_{19}/ is the following system is casual or not and why

$y[n] = 0.5\, x(n) + 3x(n - 5)$

Sol: this system is casual system the system depends on the present and past values.

7- Stable and Unstable Systems

There are several definitions for stability. Here we will consider bounded input bonded output (BIBO) stability. A system is said to be BIBO stable if every bounded input produces a bounded output. We say that a signal $\{x[n]\}$ is bounded if

$|x[n]| < M < \infty$ for all n

The moving average system

$y[n] = \frac{1}{2N+1}\sum_{n=-N}^{N} x[n]$, is stable as $y[n]$ is sum of finite numbers and so it is bounded. The accumulator system defined by

$y[n] = \sum_{k=-\infty}^{n} x[k]$. Is unstable.

If we take $\{x[n]\} = \{u[n]\}$, the unit step then

$y[0] = 1$, $y[1] = 2$, $y[2] = 3$, are $y[n] = n + 1$, $n \geq 0$ so $y[n]$ grows without bound.

❖ **System Function**

Similar to the case of the continuous-time LTI system, with the unilateral z-transform, the system function $H(z) = \frac{Y(z)}{X(z)}$ is defined under the condition that the system is relaxed, that is, all initial conditions are zero.