ENGINEERING ANALYSIS

Computer Engineering Departement

3RD stage

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Complex Number

Since there is no real number (x), which satisfies the polynomial equation $x^2 + 1 = 0$,or similar equations, the set of complex numbers is introduced.

The complex number can be written in the form of $(a + bj)$, where (a, b) are real numbers and (j) is $(\sqrt{-1})$.

Properties of Complex Number

- The two complex numbers $(a + bj)$ & $(c + dj)$ if $[a = c]$&$[b = d]$
- $(a + bj) \mp (c + dj) = (a \mp c) + (b \mp d)j$
- $(a + bj) * (c + dj) = (ac - bd) + (bc + ad)j$
- $\frac{(a+bj)}{(c+dj)} = \frac{(a+bj)*(c-dj)}{(c+dj)*(c-dj)} = \frac{(ac+bd)+(bc-ad)j}{(c^2+d^2)}$
- The complex conjugate of $(a \pm bj)$ is $(a \mp bj)$
- Absolute value of a complex number $[z = a + bj]$ is $|z| = |a + bj|$
- $= \sqrt{a^2 + b^2}$

- The distance between two complex numbers $[z_1 = a + bj]$ & $[z_2 = c + dj]$ is $|z_1 - z_2| = \sqrt{(a-c)^2 + (b-d)^2}$
- The polar form of the complex number $[a + bj]$ is $(re^{j\theta})$ where $r = \sqrt{a^2 + b^2}$, and $\theta = tan^{-1}(\frac{b}{a})$
- $e^{j\theta} = \cos(\theta) + jsin(\theta)$
- $(j)^n = \begin{cases} -1\ when\ (n) is\ even \\ 1\ when\ (n) is\ odd \end{cases}$
- If $z_1 = r_1 e^{j\theta_1}$ & $z_2 = r_2 e^{j\theta_2}$, then $z_1 . z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
- If $z_1 = r_1 e^{j\theta_1}$ & $z_2 = r_2 e^{j\theta_2}$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
- If $z = re^{j\theta}$, then $z^n = r^n e^{jn\theta}$

- If $z = re^{j\theta}$, then $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j(\frac{\theta + 2\pi k}{n})}$ or in rectangular coordinate if $z = a + bj$, then $z^{1/n} = [\ r\,(\cos\theta + J \sin\theta)\ ]^{1/n}$, where k = 0, $\pm 1, \pm 2, \ldots\ldots$

Figure (1) shows how to convert from rectangular form to polar form, which gives:

$r = \sqrt{x^2 + y^2} y$

$\theta = \tan^{-1}(\frac{y}{x})$

$x = r \cos\theta$

$y = r \sin\theta$

Not that (θ) in radian

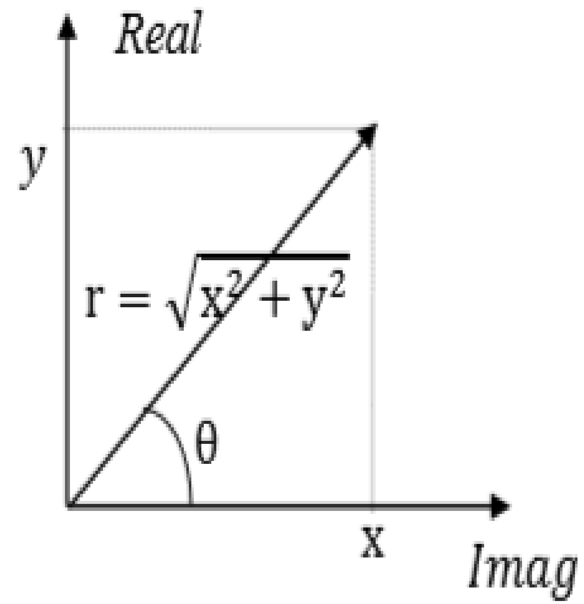


Fig (1)

❖ De . Moiver's Theorem

If $z_1 = x_1 + j\,y_1 = r_1\,(\cos\theta_1 + j\sin\theta_1) = r_1 e^{j\theta_1}$ &

$z_2 = x_2 + jy_2 = r_2\,(\cos\theta_2 + j\sin\theta_2) = r_2 e^{j\theta_2}$

Generally

$z_1 z_2 z_3 z_4 \ldots . z_n = r_1 r_2 r_3 r_4 \ldots . r_n[\cos(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \cdots \theta_n) +$

$j\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4 + \cdots \theta_n)]$

If $z_1 = z_2 = z_3 = z_4 = \cdots = z_n$, then

$z_1 z_2 z_3 z_4 \ldots . z_n == r^n[\cos(\theta n) + jsin(\theta n)$

Ex1/ for ($z_1 = 3 + j4$), ($z_2 = -2 + j1$) & ($z_3 = j$) find and plot ($z_1 z_2 z_3$)

Sol:

From De. Moiver's Theorem

$z_1 z_2 z_3 = r_1 r_2 r_3[\cos(\theta_1 + \theta_2 + \theta_3) + jsin(\theta_1 + \theta_2 + \theta_3)$

$r_1 = \sqrt{(3)^2 + (4)^2} = 5,\ \theta_1 = \tan^{-1}\left(\frac{4}{3}\right) = 53.13°$

$r_2 = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}\ ,\ \theta_2 = \tan^{-1}\left(\frac{-1}{2}\right) = -26.57°$

$r_3 = \sqrt{(0)^2 + (1)^2} = 1\ ,\ \theta_1 = \tan^{-1}\left(\frac{1}{0}\right) = 90°$

$\rightarrow$

$z_1 z_2 z_3 = 5\sqrt{5}[cos(53.13° - 26.57° + 90°) + jsin(53.13° - 26.57° + 90°)$

$= 5\sqrt{5}[cos(116.6°) + jsin(116.6°)$

$\cong -5 + j10$

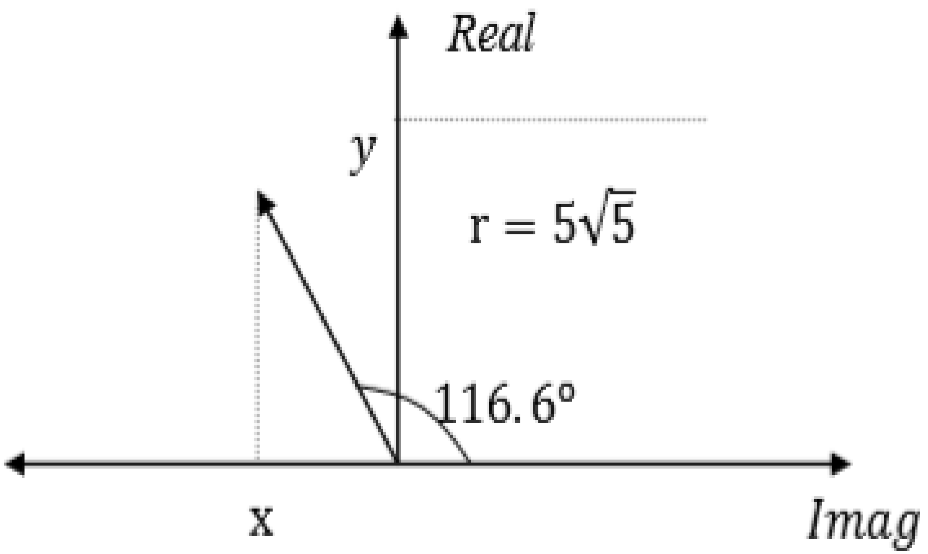


Fig (2)

Ex_2 : find $\sqrt[\frac{1}{4}]{1-j2}$

Sol:

Since $z = 1 - j2$ & $n = 4$, then $k = [0,1,2,3]$

$r = \sqrt{(1)^2 + (2)^2}$

$= \sqrt{5}$

$\theta = \tan^{-1}(\frac{-2}{1})$

$= 296.56^o$

$\cong 5.2\pi$

$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j(\frac{\theta + 2\pi k}{n})}$

$= (5)^{\frac{1}{8}}\, e^{j(\frac{5.2\pi + 2\pi k}{4})}$

If $k = 0 \rightarrow$

$z_0 = (5)^{\frac{1}{8}}\, e^{j(\frac{5.2\pi + 2\pi(0)}{4})}$

$= (5)^{\frac{1}{8}}\, e^{j\left(\frac{13}{10}\pi\right)}$

$$= (5)^{\frac{1}{8}} [\cos\left(\frac{13}{10}\pi\right) + j\sin(\frac{13}{10}\pi)$$

$$= -0.72 - j0.99$$

If $k = 1 \longrightarrow$

$$z_1 = (5)^{\frac{1}{8}} e^{j\left(\frac{5.2\pi + 2\pi(1)}{4}\right)}$$

$$= (5)^{\frac{1}{8}} e^{j\left(\frac{9}{5}\pi\right)}$$

$$= (5)^{\frac{1}{8}} [\cos\left(\frac{9}{5}\pi\right) + j\sin(\frac{9}{5}\pi)$$

$$= 0.989 - j0.72$$

If $k = 2 \longrightarrow$

$$z_2 = (5)^{\frac{1}{8}} e^{j\left(\frac{5.2\pi + 2\pi(2)}{4}\right)}$$

$$= (5)^{\frac{1}{8}} e^{j\left(\frac{23}{10}\pi\right)}$$

$$= (5)^{\frac{1}{8}} [\cos\left(\frac{23}{10}\pi\right) + j\sin(\frac{23}{10}\pi)$$

$$= 0.72 + j0.989$$

If $k = 3 \longrightarrow$

If $k = 3 \rightarrow$

$$z_3 = (5)^{\frac{1}{8}} e^{j\left(\frac{5.2\pi + 2\pi(3)}{4}\right)}$$

$$= (5)^{\frac{1}{8}} e^{j\left(\frac{14}{5}\pi\right)}$$

$$= (5)^{\frac{1}{8}} [\cos\left(\frac{14}{5}\pi\right) + j\sin(\frac{14}{5}\pi)$$

$$= -0.989 + j0.72$$

Complex Variable

If to each of a set of complex numbers which a variable z may assume there corresponds one or more values of a variable (w), then w is called a function of the complex variable (z), written $w = f(z)$.

A function is single-valued if for each value of (z) there corresponds only one value of w; otherwise, it is multiple valued or many-valued. In general, we can write $[w = f(z) = u(x,y) + iv(x,y)]$, where (u) and (v) are real functions of (x) and (y).

Ex3/ $w = z^2 = (x + jy)^2 = x^2 - y^2 + 2ixy = u + iv$ so that $(x,y) = x^2 - y^2, v(x,y) = 2xy$.

These are called the real and imaginary parts of $w = z^2$ respectively.

Ex₄/ find $(v \,\&\, v)$ in term of $(x \,\&\, y)$if $g = u^2 - j\tan^{-1}(v)$, $z = \ln(x) - jy$, and $g = z - 2$

Sol:

Since $g = z - 2 \rightarrow g = \ln(x) - jy - 2$

$\rightarrow u^2 - j\tan^{-1}(v) = \ln(x) - jy - 2$

$\therefore\ u^2 = \ln(x) - 2 \rightarrow u = \sqrt{\ln(x) - 2}$

$\tan^{-1}(v) = y \rightarrow v = \tan(y)$

Ex₅/ write the function $w = z^2 + e^z$ in the form w= m(x,y)+JU(x,y)

Sol: setting z= x+Jy

$z = f(z) = (x + jy)^2 + 2(x + jy)$

$W=x^2-y^2+J2xy+2x+2Jy$

$W=(x^2-y^2+2x)+J(2xy+2y)$

$W=(x^2-y^2+2x)+2J(xy+y)$

$u=x^2+2X-y^2$ $\&v=2(xy+y)$

Ex6/ graph the function [*w* = z2] in (*uv*) plane?

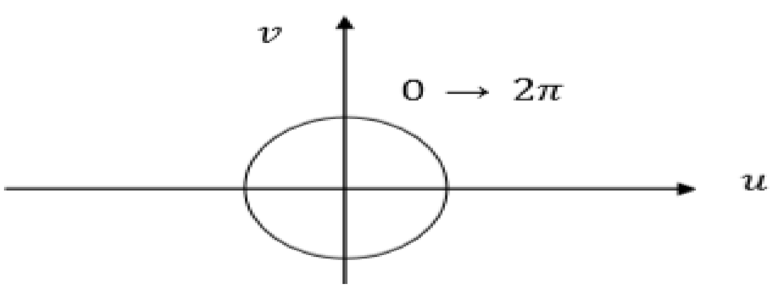
Sol: since $z = x + jy$

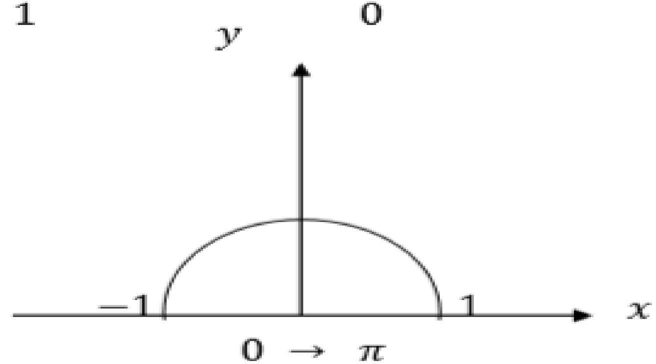
$\therefore z^2 = x^2 - y^2 + 2jxy$

$w = u(x, y) + jv(x, y)$

Where $u = x^2 - y^2$, $v = 2xy$

z		w	
x	y	u	v
1	0	1	0
$\sqrt{2}$	$\sqrt{2}$	0	4
$1/\sqrt{2}$	$1/\sqrt{2}$	0	1
0	1	-1	0
-1	0	1	0





Derivative of Complex Variable

If $w = f(z)$, $w = u + jv$, and $z = x + jy$, then

$$\frac{dw}{dx} = \frac{du}{dx} + j\frac{dv}{dx} \quad \ldots\ldots\ldots(1)$$

$$\frac{dw}{dy} = \frac{du}{dy} + j\frac{dv}{dy} \quad \ldots\ldots\ldots(2)$$

Cauchy- Riemann Conditions

The following two conditions are called Cauchy- Riemann Conditions, and these two conditions are considered two now if the function of complex variables are analytic or not.

$$\frac{du}{dx} = \frac{dv}{dy} \qquad \dots \dots \dots (3)$$

$$\frac{du}{dy} = -\frac{dv}{dx} \qquad \dots \dots \dots (4)$$

<u>Proof:</u>

Since $w = f(z)$

$w = u + jv$, and $z = x + jy$

$$\frac{dz}{dx} = 1, \frac{dz}{dy} = j$$

$$\frac{dw}{dx} = \frac{df}{dz}\frac{dz}{dx} = \frac{df}{dz} \qquad \dots \dots \dots (5)$$

From equations (1&5), it can be found that

$$\frac{df}{dz} = \frac{du}{dx} + j\frac{dv}{dx} \qquad \dots \dots \dots (6)$$

In addition:

$$\frac{dw}{dy} = \frac{df}{dz}\frac{dz}{dy} = j\frac{df}{dz} \quad \dots\dots\dots(7)$$

$$j\frac{df}{dz} = \frac{du}{dy} + j\frac{dv}{dy} \quad \dots\dots\dots(8)$$

Substitutes equation(6) in (8)

$$j\left(\frac{du}{dx} + j\frac{dv}{dx}\right) = \frac{du}{dy} + j\frac{dv}{dy}$$

$$j\frac{du}{dx} - \frac{dv}{dx} = \frac{du}{dy} + j\frac{dv}{dy} \quad \rightarrow$$

$$\frac{du}{dy} = -\frac{dv}{dx} \quad \& \quad \frac{du}{dx} = \frac{dv}{dy} \quad \dots\dots\dots(9)$$

Analyticity

The most important condition that must be considered to decide that the function is analytic or not is the Cauchy-Riemann conditions, the function must be regular within a region (R) which means that all points (z_o) in the region $(R)$is single valued in this region and have a unique finite value.

Ex$_7$/ is the following function is analytic or not?

$f(z) = \cos(z)$ where $z = x + jy$

$$f(z) = \frac{e^{jz} + e^{-jz}}{2}$$

$$= \frac{e^{j(x+jy)} + e^{-j(x+jy)}}{2}$$

$$= \frac{1}{2}(e^{jx}e^{-y} + e^{-jx}e^{y})$$

$$= \frac{1}{2}\left[e^{-y}\left(\cos(x) + j\sin(x)\right) + e^{y}\left(\cos(x) - j\sin(x)\right)\right]$$

$$= \frac{1}{2}\left[e^{-y}\cos(x) + e^{y}\cos(x) + je^{-y}\sin(x) - je^{y}\sin(x)\right]$$

$$= \frac{1}{2}\left[(e^{y} + e^{-y})\cos(x) + j(e^{-y} - e^{y})\sin(x)\right]$$

This mean

$$u = \frac{1}{2}(e^{y} + e^{-y})\cos(x)$$

$$v = \frac{1}{2}(e^{-y} - e^{y})\sin(x)$$

$$\frac{du}{dx} = \frac{-1}{2}(e^{y} + e^{-y})\sin(x)$$

$$\frac{dv}{dy} = \frac{1}{2}(-e^{-y} - e^{y})\sin(x) = \frac{-1}{2}(e^{-y} + e^{-y})\sin(x)$$

$$\frac{du}{dy} = \frac{1}{2}(e^{y} - e^{-y})\cos(x)$$

$$\frac{dv}{dx} = \frac{-1}{2}(e^{-y} - e^{y})\cos(x) = \frac{1}{2}(e^{y} - e^{-y})\cos(x)$$

Since

$$\frac{du}{dx} = \frac{dv}{dy} \quad \& \quad \frac{du}{dy} = -\frac{dv}{dx}$$

∴ The function $f(z) = \cos(z)$ is analytic function.

Ex8/ If $(Z = x + jy)$, is the function $(G = \frac{2}{z})$ analytic or not?

Sol:

$$G = \frac{2}{Z}$$

$$= \frac{2}{x + jy} \cdot \frac{x - jy}{x - jy}$$

$$= \frac{2x - j2y}{x^2 + y^2} \rightarrow$$

$$u = \frac{2x}{x^2 + y^2} \quad \& \; v = \frac{2y}{x^2 + y^2}$$

$$\frac{du}{dx} = \frac{2(x^2 + y^2) - 2x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{dv}{dy} = \frac{-2(x^2 + y^2) + 2y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-2x^2 - 2y^2 + 2y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

And in the same way

$$\frac{du}{dv} = \frac{-2x(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-4xy}{(x^2 + y^2)^2}$$

$$\frac{dv}{dx} = \frac{-2y(2x)}{(x^2 + y^2)^2}$$

$$= \frac{-4yx}{(x^2 + y^2)^2}$$

$$\therefore \frac{du}{dx} = \frac{dv}{dy} \quad \& \quad \frac{du}{dy} = -\frac{dv}{dx}$$

This means that the Cauchy-Riemann conditions are satisfied but not only for all points, for example the point (0, 0)

At $y = 0 \rightarrow u = \frac{1}{x}$ and $\frac{du}{dx} = \frac{-1}{x^2}$

Similarly at $x = 0 \rightarrow v = \frac{-1}{y}$ and $\frac{dv}{dy} = \frac{-1}{y^2}$

$$\frac{du}{dx} \neq \frac{dv}{dy} \quad \& \quad \frac{du}{dy} = -\frac{dv}{dx} = 0$$

∴The Cauchy-Riemann conditions are not fully satisfied, since the function ($G = \frac{2}{z}$) is not analytic at point ($Z = 0$)

THANK YOU FOR YOUR LESITENING

ANY Q ?