ENGINEERING ANALYSIS
Computer Engineering Departement
3RD stage

L 12

ASS.L.RAGHDA SALAM ALI

COMPLEX INTEGRATION

In this section, two types of integration will be illustrated

1- Line integral

Which is denoted by $\int_c f(z)dz$

Or $\int_a^b f(z)dz$

For $f(z) = u + jv$ & $z = x + jy \rightarrow dz = dx + jdy$

$$\rightarrow \int_c f(z)dz = \int_c (u + jv)(dx + jdv)$$

$$\int_c f(z)dz = \int_c [(udx - vdy) + j(udy + vdx)]$$

Ex$_{10}$/ find ($\int_c e^z\, dz$) from [$z = 0$ to $z = 2 + j2$]

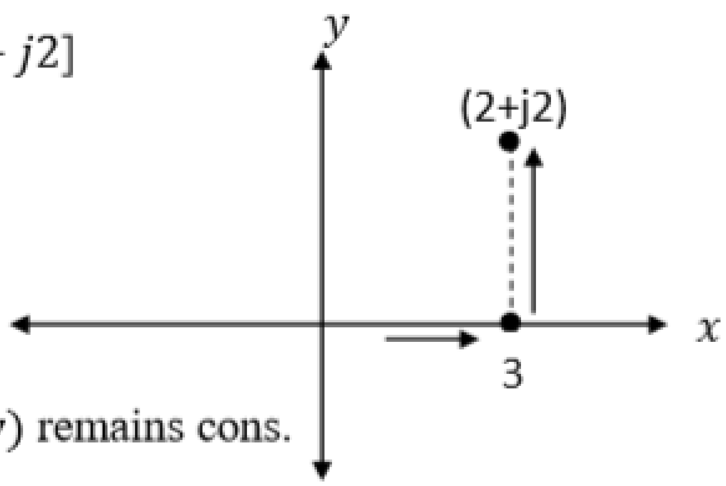


Fig (4)

Sol:

First path:

From $z = 0 + j0$ to $z = 3 + j0$

In this path (x) changes from (0) to (2) & (y) remains cons.

$e^z = e^{x+jy}$

$= e^x[\cos(y) + j\sin(y)]$

$= e^x\cos(y) + je^x\sin(y) \rightarrow u = e^x\cos(y)$ & $v = e^x\sin(y)$

$x = (0\ to\ 2)$ & $y = 0 \rightarrow$

$I = \int_0^2[e^x\cos(0)\,dx - e^x\sin(0)\,dy] + j\int_0^2[e^x\cos(0)dy + e^x\sin(0)dx]$

$=\int_0^2 e^x\, dx$

$=[e^x]_0^2$

$=(e^2-1)$

For the second path:

$y=(0\ to\ 2)\ \&\ x=2 \rightarrow$

$I=\int_0^2[e^2\cos(y)\,dx-e^2\sin(y)\,dy]+j\int_0^2[e^2\cos(y)dy+e^2\sin(y)dx]$

$=-\int_0^2 e^2\sin(y)\,dy+j\int_0^2 e^2\cos(y)dy$

$=e^2[\cos(y)]_0^2+je^2[\sin(y)]_0^2$

$=e^2\cos(2)-e^2\cos(0)+je^2\sin(2)-e^2\sin(0)$

$=\ e^2[\cos(2)+j\sin(2)]-e^2$

$=\ e^2\left(\ e^{j2}-1\right)$

$\therefore I_t=\ (e^2-1)+\ e^2\left(\ e^{j2}-1\right)$

$=e^{j4}-1$

Ex11/ finc $\int_c (z^2-2)dz$, if (z)changes from $(-1+j1)$ to $(2+j1)$ & $(y=2x)$

Sol:

$f(z)=z^2-2$

$=(x^2-y^2+2jxy) \rightarrow u=x^2-y^2 \;\&\; v=\;2xy$

$\rightarrow I=\int_c[(x^2-y^2)dx-\;2xydy]+j\int_c[(x^2-y^2)dy+\;2xydx]$

Since $(y=2x\;\rightarrow dy=2dx)$, then

$$I=\int_{-1}^{2}[(x^2-4x^2)dx-\;8x^2dx]+j\int_{-1}^{2}[2(x^2-4x^2)dx+\;4x^2dx]$$

$=\frac{11}{3}\,[x^3]_{-1}^{2}+j\,\frac{2}{3}\,[x^3]_{-1}^{2}$

$=\frac{11}{3}\,[8+1]+j\frac{2}{3}\,[8+1]$

$=33+j\,6$

Change of variable

Let $z = g(\alpha)$ be a continuous function of a complex variable $\alpha = u + jv$ suppose that the curve (c)in the (z) plane corresponding to curve $(\bar{c})$ with the (α)plane & that the derivative $\bar{g}(\alpha)$ is continuo's on $(\bar{c})$ then :

$$\int_c f(z)dz = \int_{\bar{c}} f\{g(\alpha)\}\bar{g}(\alpha)d\alpha$$

Ex12:- evaluate $\int_c z\,dz$ from $z = 0$ to $z = 4 + j2$ along the curve c given by $z = t^2 + jt$?

Sol:- here $t = \alpha$

$g(t) = t^2 + jt \implies \bar{g}(t) = 2t + j$

when $z = 0$

Since $z = t^2 + jt \implies t = 0$

$Z = 4 + 2j$

$\therefore\ 4 + 2j = t^2 + jt \implies t = 2$

$\therefore \int_c zdz = \int_0^2 (t^2 + jt)(2t + j)dt$

$= 6 + 8j$

Remark1: when the direction of the counter integration is changed the the sign of integration changes too.

Remark2: Simple closed curve, simple & multiply connected region.

A curve is called a simple closed curve if does not cross it self figure (5.1) is a simple closed curve while figure (5.2) is not simple closed & is known as multiple curve .

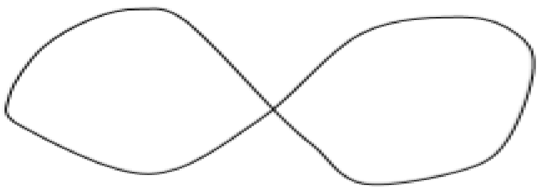


Fig 5.2

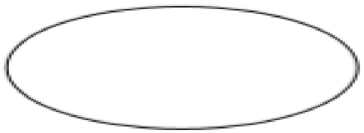
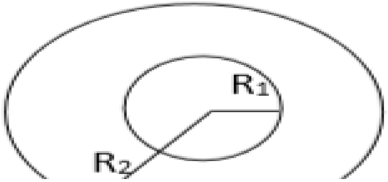


Fig 5.1

<u>Remark3:</u> Aregion is called simple connected if every closed curve in the region enclosed point of the region only . aregion which is not simple connected is called multiply connected for example :The region between two concentric circuit $r_1 \leq |z - z_0| \leq r_2$ as shown in figure (6) is an example of amultiple connected region .



❖ Cauchy's theorem of integral

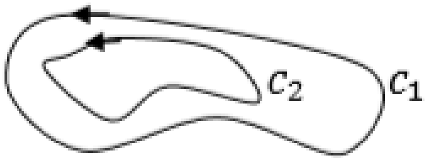
This theorem states that, when $f\ (z)$ is analytic & $\overline{f}(z)$ is continuous inside & on simple closed curve, then:

$$\oint_c f(z)dz = 0 \ldots\ldots\ldots (12)$$

This integration is called a contour integration.

If $f(z)$ is analytic in the doubly connected region bounded by the curve $c_1 \& c_2$ as illustrated in figure (7) then:

$$\oint_{c_1} f(z)dz = \oint_{c_2} f(z)dz \ \ldots\ldots\ldots (13)$$



Fig(7)

Ex13 / evaluate [$\oint_{c_1} \frac{dz}{z-a}$], where (c) is any simple closed curve &$(z = a)$ is

1- outside the curve (c)
2- inside the curve (c)

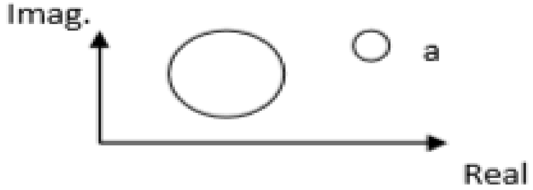


Fig (8)

Sol:

1- outside the curve (c)

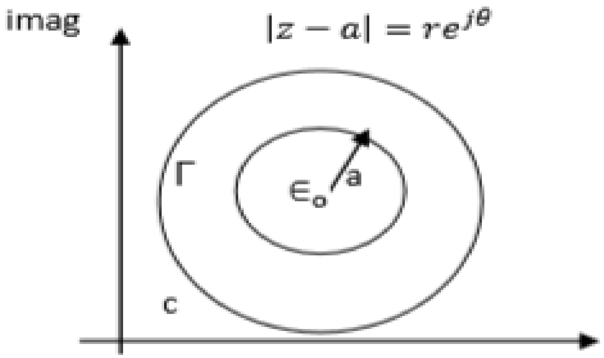
$$\oint_c \frac{dz}{z-a} = 0$$

2- inside the curve (c)

Let be a Circular of radius (ϵ_o), with center at $Z = a$ so that (Γ) is inside (c) , recall Cauchy's theorem for multiply connected Region

$$\oint_{c_1} f(z)dz = \oint_{c_2} f(z)dz$$

$$\oint_c \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a}$$



$$|z-a| = \epsilon_o$$

$$z-a = \epsilon_o e^{j\theta}, (z = e^{j\theta} + a)$$

diff. both side

$$dz = j\,\epsilon_o e^{j\theta} d\theta$$

For counter integration around the circuit of radius (ϵ_o), lower limit $(\theta = 0)$ & upper limit $(\theta = 2\pi)$

$$\therefore \oint_{\Gamma} \frac{dz}{z-a} = \oint_{\Gamma} \frac{j\epsilon e^{j\theta} d\theta}{\epsilon e^{j\theta}+a-a} = j\oint_0^{2\pi} d\theta = 2\pi j = \oint_c \frac{dz}{(z-a)}$$

Cauchy's integral formula

If $f(z)$ analyticall inside & on asimple closed curve (c & z_o) is any point inside (c) then :

$$f(z_o) = \frac{1}{2\pi j} \oint_c \frac{f(z)}{z-z_0}\, dz \quad \ldots\ldots\ldots (14)$$

or
$$\oint_c \frac{f(z)}{z-z_0}\, dz = 2\pi j\, f(z_o) \quad \ldots\ldots\ldots (15)$$

The derivatives of (n) order for $f(z)$ at $(z = z_o)$ is given by

$$f^n(z_o) = \frac{n!}{2\pi j} \oint_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \ldots\ldots\ldots (16)$$

where n =1,2,3,…….order of derivative .

Ex14: evaluate $\oint_c \frac{e^{-z}}{(z-3)(z-2)}\, dz$, where (c) is the circle, $|z| = 3$?

Sol:

Decompose the denominator

$$\frac{1}{(z-3)(z-2)} = \frac{B}{z-3} + \frac{A}{z-2}$$

$\rightarrow$

$$\frac{Bz - 2B + Az - 3A}{(z-3)(z-2)} = \frac{1}{(z-3)(z-2)}$$

$$\rightarrow Bz - 2B + Az - 3A = 1$$

$$\rightarrow \quad B + A = 0 \quad \dots\dots\dots(17)$$

$$\underline{2B - 3A = 1} \quad \dots\dots\dots(18)$$

Multiply equation (17) by (3) and adding the two equations (17),(18)

$\rightarrow \quad B + 3A = 0 \quad \dots\dots\dots (19)$

$\underline{2B - 3A = 1} \quad \dots\dots\dots (20)$

$3B = 1 \rightarrow B = \frac{1}{3}$ substitutes in equation (17) then

$\frac{1}{3} + A = 0 \rightarrow A = -\frac{1}{3}$

$\therefore \oint_c \frac{e^{-z}}{(z-3)(z-2)} dz = \frac{1}{3}\left(\oint_c \frac{e^{-z}}{(z-3)} dz - \oint_c \frac{e^{-z}}{(z-2)} dz\right)$

According to the Caushys integral formula

$\oint_c \frac{e^{-z}}{(z-3)} dz = 2\pi\, j\, f(3)$

$= 2\pi\, j\, e^{-3}$

& $\oint_c \frac{e^{-z}}{(z-2)} dz = 2\pi\, j\, e^{-2}$

This means that $\oint_c \frac{e^{-z}}{(z-3)(z-2)} dz = \frac{1}{3}[2\pi\, j\, e^{-3} + 2\pi\, j\, e^{-2}]$

Ex15: evaluate $\oint_c \frac{\cos(2z)}{(z+\pi)^3}$ where (c) is the circle |z|=3?

Sol:

By Cauchy's formula

$$f^n(z_o) = \frac{n!}{2\pi j} \oint_c \frac{f(z)}{(z-z_o)^{n+1}} \mathrm{dz}$$

$$n+1 = 3 \Longrightarrow n = 2$$

$$f(z) = \cos(2z) \rightarrow \bar{f}(z) = -2\sin(2z), \bar{\bar{f}}(z) = -4\cos(2z)$$

$$\bar{\bar{f}}(z) = -4\cos(2z) \rightarrow \bar{\bar{f}}(\pi) = -4\cos(2\pi)$$

$$\bar{\bar{f}}(\pi) = -4$$

$$\rightarrow$$

$$\bar{\bar{f}}(\pi) = \frac{2!}{2\pi j} \oint_c \frac{f(z)}{(z-z_o)^{n+1}} \mathrm{dz}$$

$$\therefore \oint_c \frac{f(z)}{(z-z_o)^{n+1}} dz = \frac{2}{2!} \pi j * -4$$

$$= -4\pi j$$

Ex16/ if (c) is a circle $[|z+2|=3]$, evaluate $\oint_c \frac{z^2}{z^2+1}\,dz$

Sol:

Since $|z+2|$ is a circle that centred at $(-2-j0)$

and has raduis $=3$

$z^2+1=0$

$z^2=-1 \rightarrow z=\pm j$

The two points $[z=j,\ z=-j]$ are located inside the circle therefore

1-) $\oint_c \frac{z^2}{z^2+1}\,dz = 2\pi j\, f(-j)$

$= 2\pi j\,(-j)^2$

$= -2\,\pi j$

2-) $\oint_c \frac{z^2}{z^2+1}\,dz = 2\pi j\, f(j)$

$= 2\pi j\,(j)^2$

$= -2\,\pi j$

$$\therefore \oint_c \frac{z^2}{z^2+1}\,dz = -2\,\pi j - 2\,\pi j$$

$= -4\pi j$

Remark4: if $f(z)$ is analytic at any point except at $(z = zo)$, then this point is called pole which is divided into [simple pole & pole of order (n)]

Remark5: when the analytic function has poles, then such function have a singularity at these poles.

Remark6: if the pole of analytic function can be removed by taking [$\lim_{z \to zo} f(z)$], then this singularity is called removable singularity.

Remark7: if the pole cannot be removed, then this singularity is called (essential singularity) .

Remark8: the analytic function is called entire function if this function is analytic at all points except at $(\pm\infty)$

Ex$_{17}$/ classify each of the following functions

1-) $f(z) = \frac{z-3}{(z-1)(z+3)}$ 2-) $f(z) = \frac{e^{-z}}{(z-8)^4}$ 3-) $f(z) = \frac{1}{e^{-\frac{1}{z}}}$

4-) $f(z) = \cos(z)$

Sol:

1- For the first function $f(z) = \frac{z-3}{(z-1)(z+3)}$

This function is analytic with simple poles $(1, -3)$ and has removable singularity

$$\lim_{z\to 1} \frac{z-3}{(z-1)(z+3)} \text{ Apply H.R} \longrightarrow \lim_{z\to 1} \frac{-3}{(-1)(3)} = 1$$

Moreover, $\lim_{z\to -3} \frac{z-3}{(z-1)(z+3)}$ Apply H.R $\longrightarrow \lim_{z\to 1} \frac{-3}{(-1)(3)} = 1$

2- For the function $f(z) = \frac{e^{-z}}{(z-8)^4}$

This function is analytic with pole (8) of order (4) and has removable singularity

$$\lim_{z\to 8} \frac{e^{-z}}{(z-8)^4} \text{ Apply H.R} \to \lim_{z\to 8} \frac{-e^{-z}}{4(z-8)^3} \text{ Apply H.R} \to \lim_{z\to 8} \frac{e^{-z}}{12(z-8)^2}$$

$$\text{Apply H.R} \to \lim_{z\to 8} \frac{-e^{-z}}{24(z-8)^1} \text{ Apply H.R} \lim_{z\to 8} \frac{e^{-z}}{24} = \frac{e^{-8}}{24}$$

3- For the function $f(z) = \frac{1}{e^{-\frac{1}{z}}} = e^{\frac{1}{z}}$

This function is analytic with pole (0) and has essential singularity

4- For the function $f(z) = \cos(z)$

This function is analytic on all the field except at $(\pm\infty)$ then it is called entire function.

Thank you for your listenning

Any Q ?