Condition 3: $|a_0| < a_2$

$$|0.368 + 0.066K| < 1 \tag{7.74}$$

For marginal stability

$$\begin{aligned} 0.368 + 0.066K &= 1 \\ K &= \frac{1 - 0.368}{0.066} = 9.58 \end{aligned} \tag{7.75}$$

Hence the system is marginally stable when $K = 9.58$ and 105.23 (see also Example 7.6 and Figure 7.20).

7.6.3 Root locus analysis in the *z*-plane

As with the continuous systems described in Chapter 5, the root locus of a discrete system is a plot of the locus of the roots of the characteristic equation

$$1 + GH(z) = 0 \tag{7.76}$$

in the z-plane as a function of the open-loop gain constant K. The closed-loop system will remain stable providing the loci remain within the unit circle.

7.6.4 Root locus construction rules

These are similar to those given in section 5.3.4 for continuous systems.

1. *Starting points* ($K = 0$): The root loci start at the open-loop poles.
2. *Termination points* ($K = \infty$): The root loci terminate at the open-loop zeros when they exist, otherwise at ∞.
3. *Number of distinct root loci*: This is equal to the order of the characteristic equation.
4. *Symmetry of root loci*: The root loci are symmetrical about the real axis.
5. *Root locus locations on real axis*: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
6. *Breakaway points*: The points at which a locus breaks away from the real axis can be found by obtaining the roots of the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\{GH(z)\} = 0$$

7. *Unit circle crossover*: This can be obtained by determining the value of K for marginal stability using the Jury test, and substituting it in the characteristic equation (7.76).

Example 7.6 (See also Appendix 1, *examp76.m*)
Sketch the root locus diagram for Example 7.4, shown in Figure 7.14. Determine the breakaway points, the value of K for marginal stability and the unit circle crossover.

Solution
From equation (7.43)

$$G(s) = K\left(1 - \frac{e^{-Ts}}{s}\right)\left\{\frac{1}{s(s+2)}\right\} \tag{7.77}$$

and from equation (7.53), given that $T = 0.5$ seconds

$$G(z) = K\left(\frac{0.092z + 0.066}{z^2 - 1.368z + 0.368}\right) \tag{7.78}$$

Open-loop poles

$$z^2 - 1.368z + 0.368 = 0 \tag{7.79}$$

$$\begin{aligned} z &= 0.684 \pm 0.316 \\ &= 1 \text{ and } 0.368 \end{aligned} \tag{7.80}$$

Open-loop zeros

$$0.092z + 0.066 = 0$$

$$z = -0.717 \tag{7.81}$$

From equations (7.67), (7.68) and (7.69) the characteristic equation is

$$z^2 + (0.092K - 1.368)z + (0.368 + 0.066K) = 0 \tag{7.82}$$

Breakaway points: Using Rule 6

$$\frac{d}{dz}\{GH(z)\} = 0$$

$$(z^2 - 1.368z + 0.368)K(0.092) - K(0.092z + 0.066)(2z - 1.368) = 0 \tag{7.83}$$

which gives

$$\begin{aligned} 0.092z^2 + 0.132z - 0.1239 &= 0 \\ z &= 0.647 \text{ and } -2.084 \end{aligned} \tag{7.84}$$

K for marginal stability: Using the Jury test, the values of K as the locus crosses the unit circle are given in equations (7.75) and (7.73)

$$K = 9.58 \text{ and } 105.23 \tag{7.85}$$

Unit circle crossover: Inserting $K = 9.58$ into the characteristic equation (7.82) gives

$$z^2 - 0.487z + 1 = 0 \tag{7.86}$$

The roots of equation (7.86) are

$$z = 0.244 \pm j0.97 \tag{7.87}$$

or

$$z = 1\angle \pm 75.9^\circ = 1\angle \pm 1.33\,\text{rad} \tag{7.88}$$

Since from equation (7.63) and Figure 7.16

$$z = |z| \angle \omega T \tag{7.89}$$

and $T = 0.5$, then the frequency of oscillation at the onset of instability is

$$\begin{aligned} 0.5\omega &= 1.33 \\ \omega &= 2.66\,\text{rad/s} \end{aligned} \tag{7.90}$$

The root locus diagram is shown in Figure 7.20.

It can be seen from Figure 7.20 that the complex loci form a circle. This is usually the case for second-order plant, where

$$\begin{aligned} \text{Radius} &= \sum |\text{open-loop poles}| \\ \text{Centre} &= (\text{Open-loop zero, } 0) \end{aligned} \tag{7.91}$$

The step response shown in Figure 7.15 is for $K = 1$. Inserting $K = 1$ into the characteristic equation gives

$$z^2 - 1.276z + 0.434 = 0$$

or

$$z = 0.638 \pm \mathrm{j}0.164$$

This position is shown in Figure 7.20. The K values at the breakaway points are also shown in Figure 7.20.

7.7 Digital compensator design

In sections 5.4 and 6.6, compensator design in the s-plane and the frequency domain were discussed for continuous systems. In the same manner, digital compensators may be designed in the z-plane for discrete systems.

Figure 7.13 shows the general form of a digital control system. The pulse transfer function of the digital controller/compensator is written

$$\frac{U}{E}(z) = D(z) \tag{7.92}$$

and the closed-loop pulse transfer function become

$$\frac{C}{R}(z) = \frac{D(z)G(z)}{1 + D(z)GH(z)} \tag{7.93}$$

and hence the characteristic equation is

$$1 + D(z)GH(z) = 0 \tag{7.94}$$