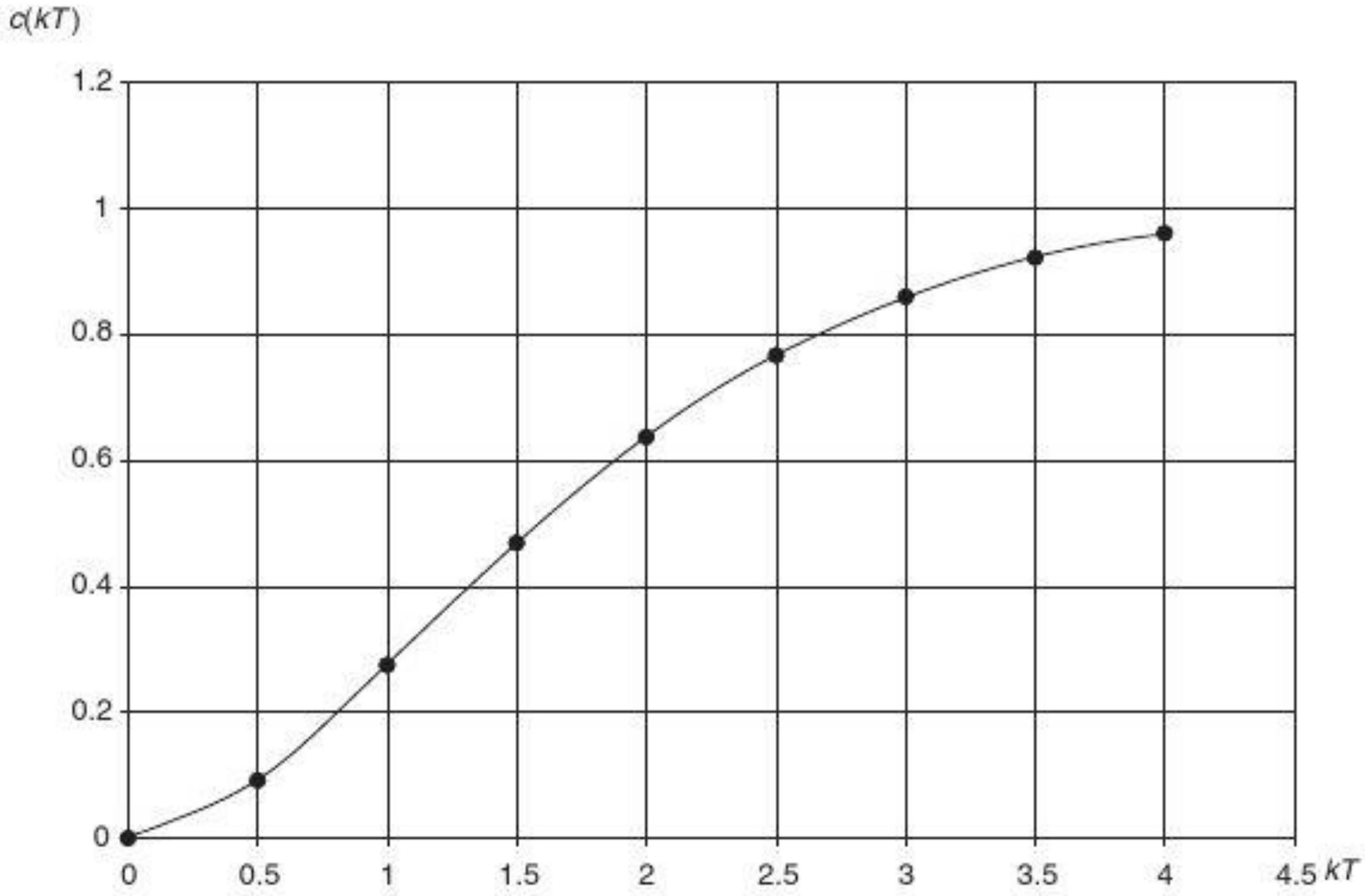

Fig. 7.15 Unit step response for Example 7.4.

$$c(\infty) = \left(\frac{0.092 + 0.066}{1 - 1.276 + 0.434}\right) = 1.0 \tag{7.60}$$

Hence there is no steady-state error.

7.6 Stability in the *z*-plane

7.6.1 Mapping from the *s*-plane into the *z*-plane

Just as transient analysis of continuous systems may be undertaken in the s-plane, stability and transient analysis on discrete systems may be conducted in the z-plane.

It is possible to map from the s to the z-plane using the relationship

$$z = e^{sT} \tag{7.61}$$

now

$$s = \sigma \pm j\omega$$

therefore

$$z = e^{(\sigma \pm j\omega)T} = e^{\sigma T} e^{j\omega T} \quad \text{(using the positive } j\omega \text{ value)} \tag{7.62}$$

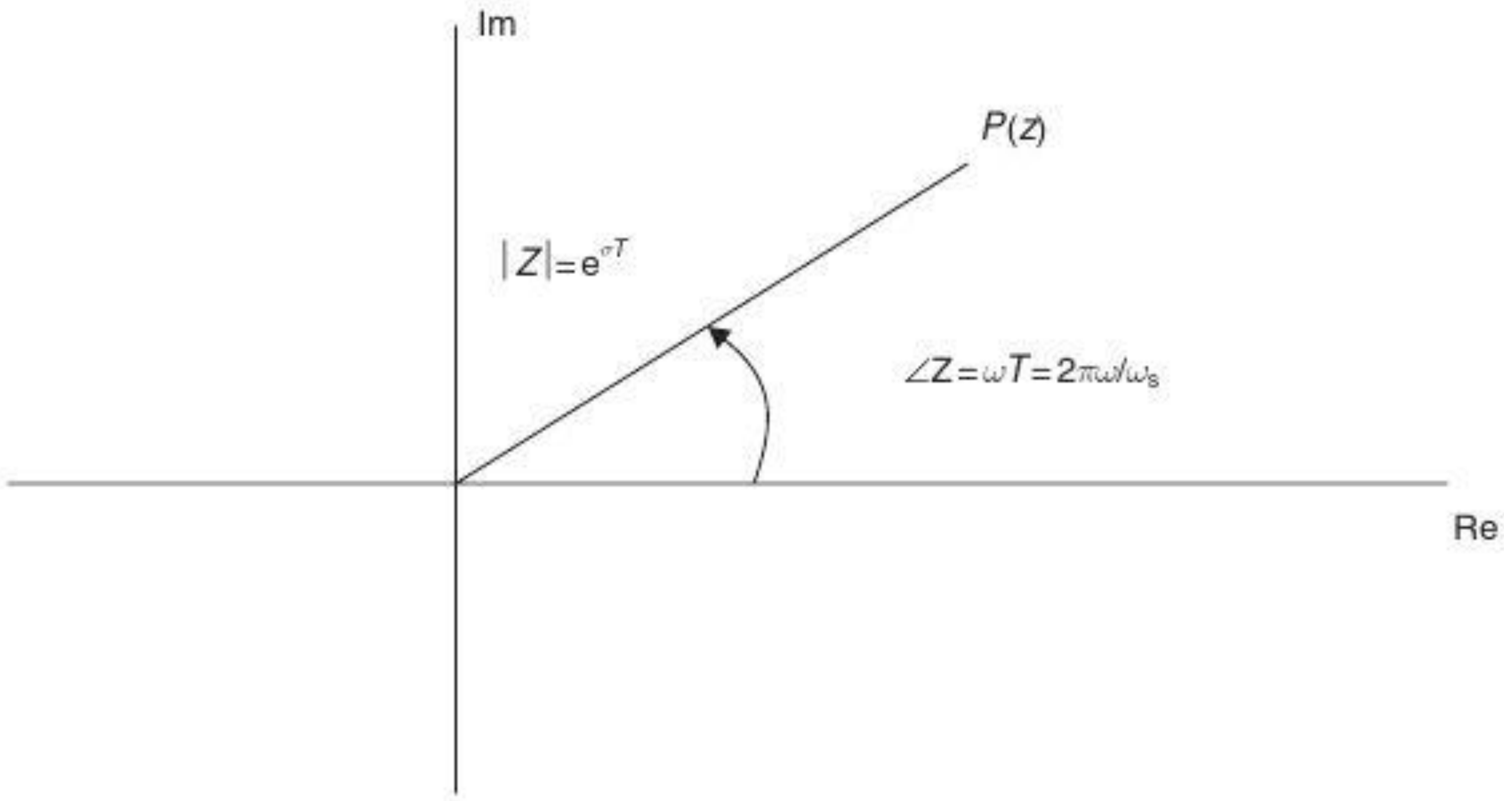

Fig. 7.16 Mapping from the *s* to the *z*-plane.

If $e^{\sigma T} = |z|$ and $T = 2\pi/\omega_s$ equation (7.62) can be written

$$z = |z|e^{j(2\pi\omega/\omega_s)} \tag{7.63}$$

where ω_s is the sampling frequency.

Equation (7.63) results in a polar diagram in the z-plane as shown in Figure 7.16. Figure 7.17 shows mapping of lines of constant σ (i.e. constant settling time) from the s to the z-plane. From Figure 7.17 it can be seen that the left-hand side (stable) of the s-plane corresponds to a region within a circle of unity radius (the unit circle) in the z-plane.

Figure 7.18 shows mapping of lines of constant ω (i.e. constant transient frequency) from the s to the z-plane.

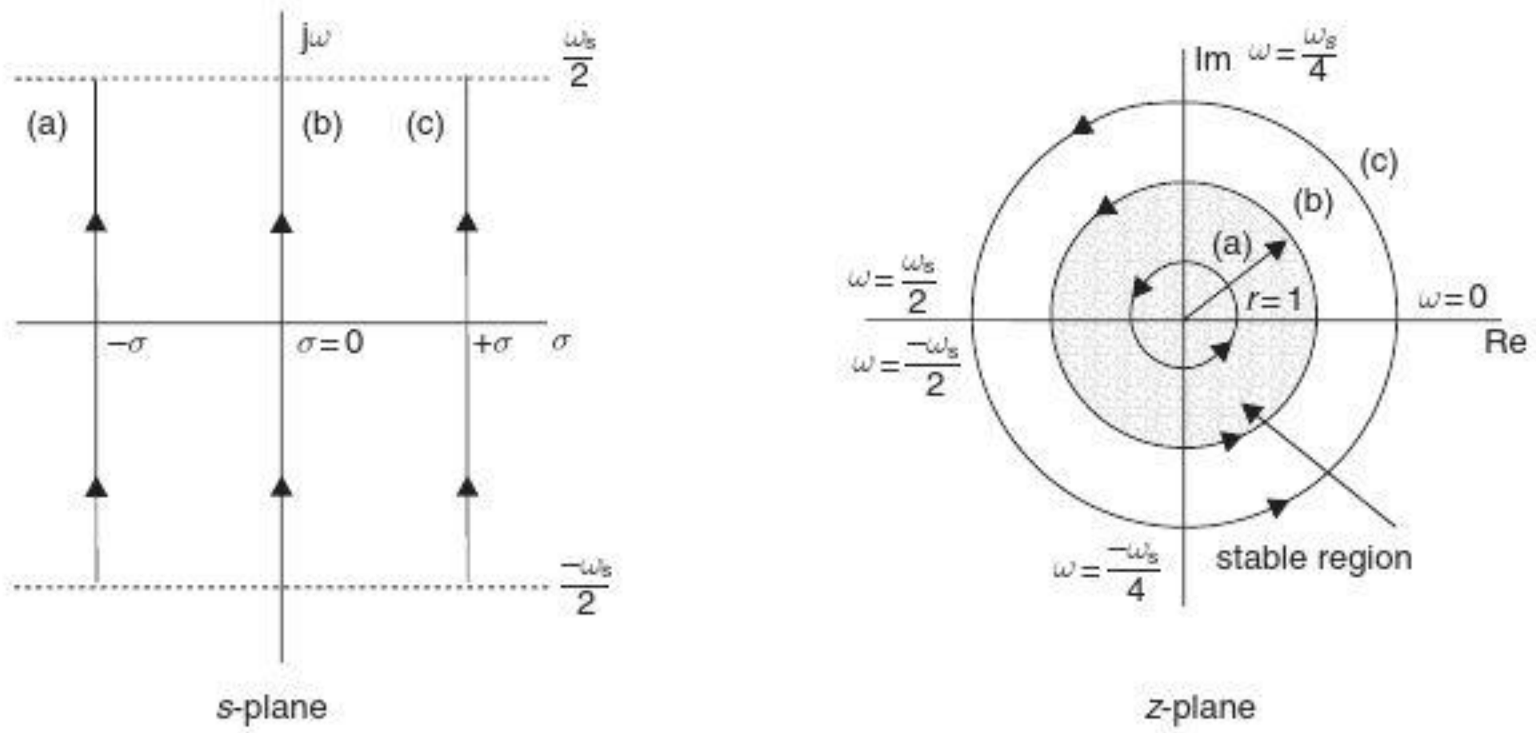

Fig. 7.17 Mapping constant σ from s to z-plane.

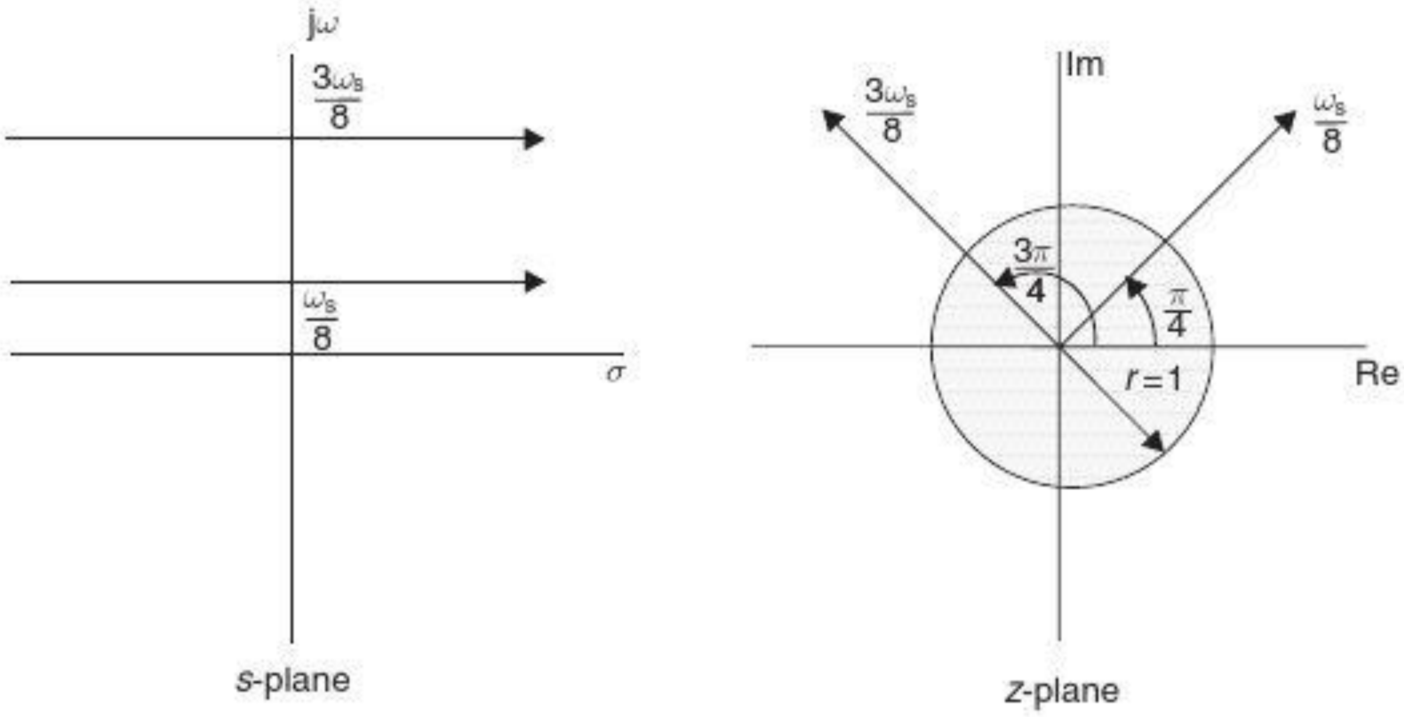

Fig. 7.18 Mapping constant ω from s to z-plane.

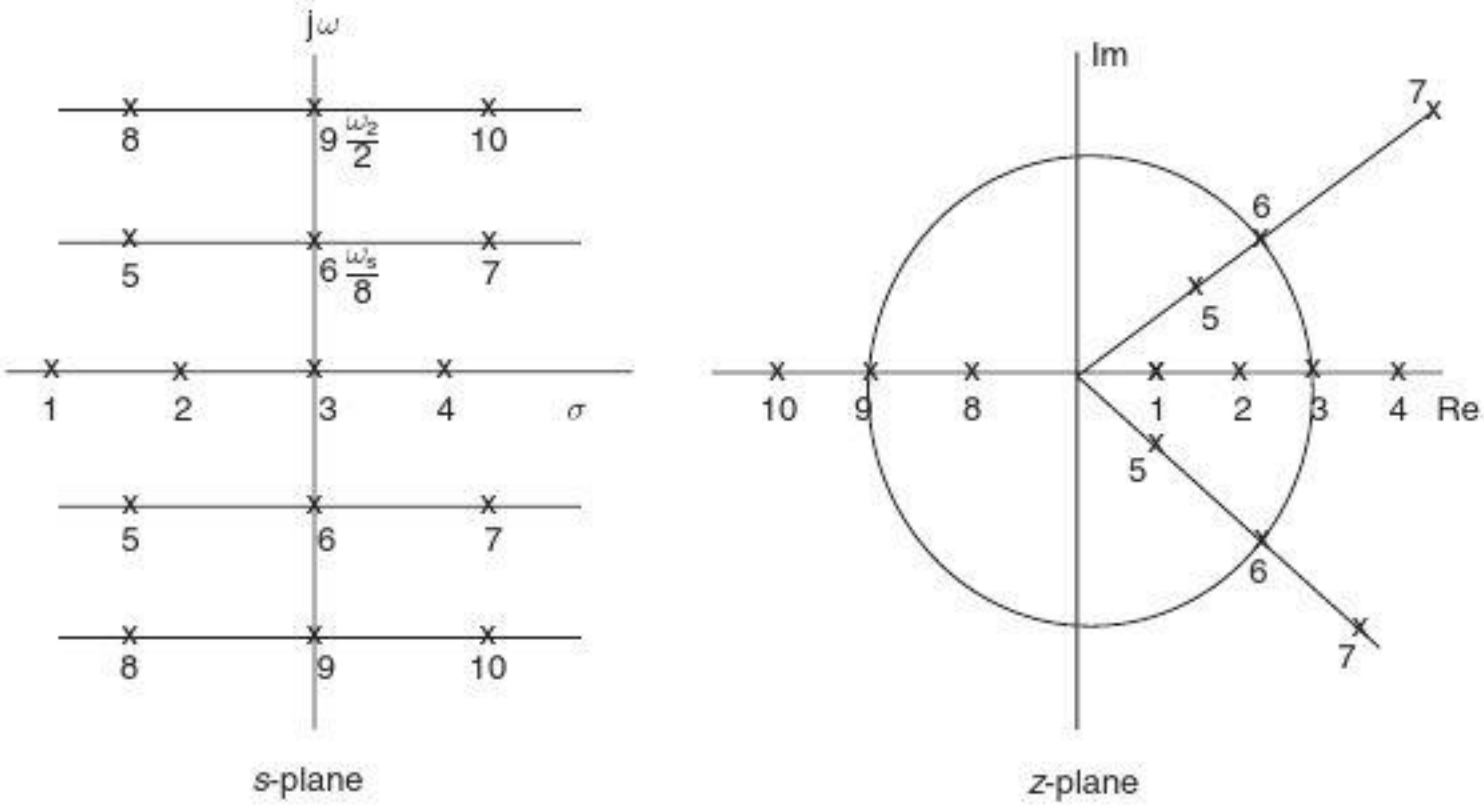

Fig. 7.19 Corresponding pole locations on both s and z-planes.

Figure 7.19 shows corresponding pole locations on both the s-plane and z-plane.

7.6.2 The Jury stability test

In the same way that the Routh–Hurwitz criterion offers a simple method of determining the stability of continuous systems, the Jury (1958) stability test is employed in a similar manner to assess the stability of discrete systems.

Consider the characteristic equation of a sampled-data system

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \tag{7.64}$$

Table 7.4 Jury's array

z^0	z^1	z^2		z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_0	
.					
.					
.					
l_0	l_1	l_2	$\dots$	l_3	
l_3	l_2	l_1	$\dots$	l_0	
m_0	m_1	m_2			
m_2	m_1	m_0			

The array for the Jury stability test is given in Table 7.4 where

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix} \tag{7.65}$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

The necessary and sufficient conditions for the polynomial $Q(z)$ to have no roots outside or on the unit circle are

$$\begin{array}{ll} \text{Condition 1} & Q(1) > 0 \\ \text{Condition 2} & (-1)^n Q(-1) > 0 \\ \text{Condition 3} & |a_0| < a_n \\ \quad\vdots & |b_0| > |b_{n-1}| \\ & |c_0| > |c_{n-2}| \\ & \quad\vdots \\ \text{Condition } n & |m_0| > |m_2| \end{array} \tag{7.66}$$

Example 7.5 (See also Appendix 1, *examp75.m*)
For the system given in Figure 7.14 (i.e. Example 7.4) find the value of the digital compensator gain K to make the system just unstable. For Example 7.4, the characteristic equation is

$$1 + G(z) = 0 \tag{7.67}$$

In Example 7.4, the solution was found assuming that $K = 1$. Therefore, using equation (7.53), the characteristic equation is

$$1 + \frac{K(0.092z + 0.066)}{(z^2 - 1.368z + 0.368)} = 0 \tag{7.68}$$

or

$$Q(z) = z^2 + (0.092K - 1.368)z + (0.368 + 0.066K) = 0 \tag{7.69}$$

The first row of Jury's array is

$$\begin{array}{c|ccc} & z^0 & z^1 & z^2 \\ \hline & (0.368 + 0.066K) & (0.092K - 1.368) & 1 \end{array} \tag{7.70}$$

Condition 1: $Q(1) > 0$
From equation (7.69)

$$Q(1) = \{1 + (0.092K - 1.368) + (0.368 + 0.066K)\} > 0 \tag{7.71}$$

From equation (7.71), $Q(1) > 0$ if $K > 0$.

Condition 2 $(-1)^n Q(-1) > 0$
From equation (7.69), when $n = 2$

$$(-1)^2 Q(-1) = \{1 - (0.092K - 1.368) + (0.368 + 0.066K)\} > 0 \tag{7.72}$$

Equation (7.72) simplifies to give

$$2.736 - 0.026K > 0$$

or

$$K < \frac{2.736}{0.026} = 105.23 \tag{7.73}$$

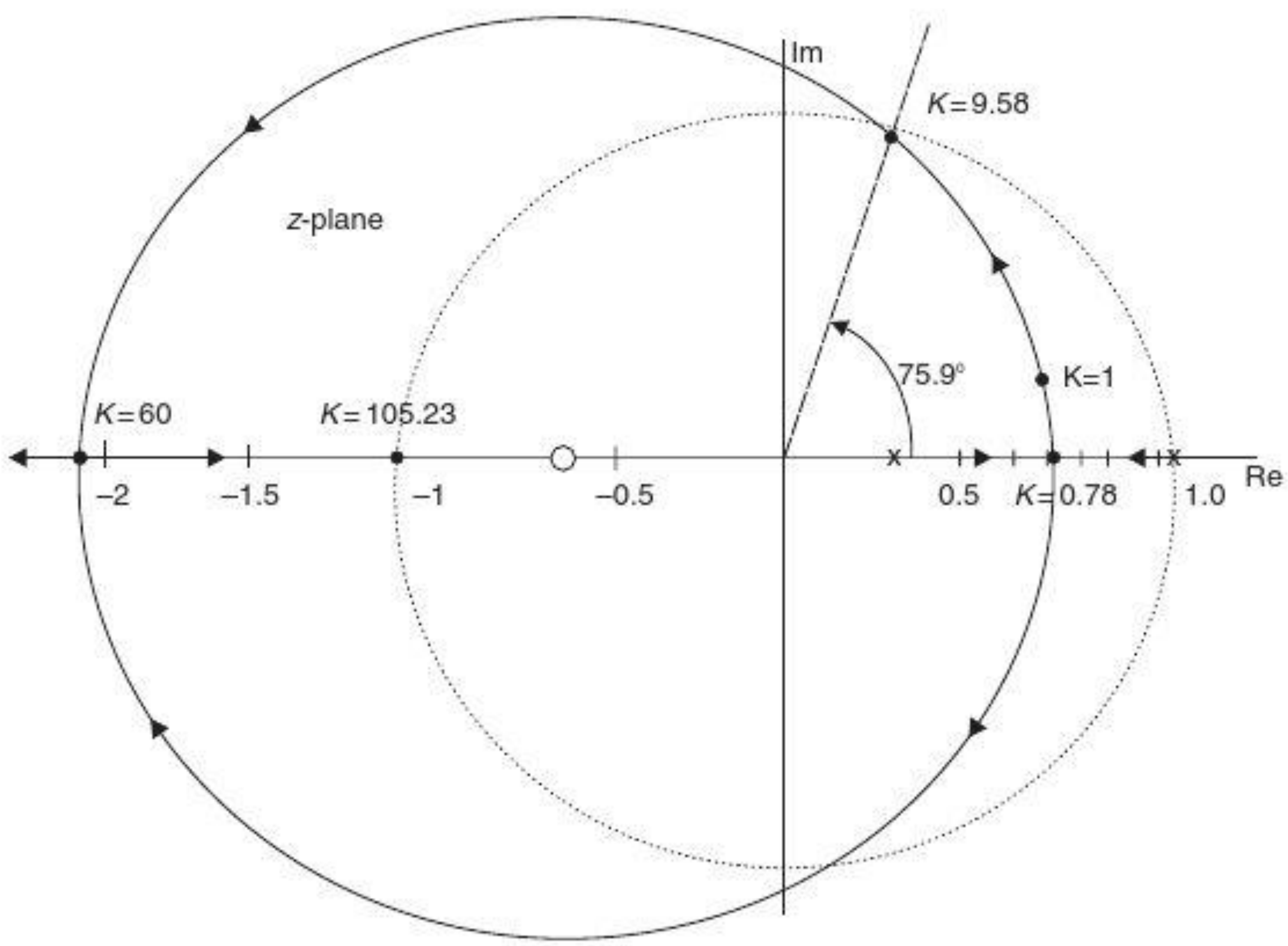

Fig. 7.20 Root locus diagram for Example 7.4.

Condition 3: $|a_0| < a_2$

$$|0.368 + 0.066K| < 1 \tag{7.74}$$

For marginal stability

$$\begin{aligned} 0.368 + 0.066K &= 1 \\ K &= \frac{1 - 0.368}{0.066} = 9.58 \end{aligned} \tag{7.75}$$

Hence the system is marginally stable when $K = 9.58$ and 105.23 (see also Example 7.6 and Figure 7.20).

7.6.3 Root locus analysis in the *z*-plane

As with the continuous systems described in Chapter 5, the root locus of a discrete system is a plot of the locus of the roots of the characteristic equation

$$1 + GH(z) = 0 \tag{7.76}$$

in the z-plane as a function of the open-loop gain constant K. The closed-loop system will remain stable providing the loci remain within the unit circle.

7.6.4 Root locus construction rules

These are similar to those given in section 5.3.4 for continuous systems.

1. *Starting points* ($K = 0$): The root loci start at the open-loop poles.
2. *Termination points* ($K = \infty$): The root loci terminate at the open-loop zeros when they exist, otherwise at ∞.
3. *Number of distinct root loci*: This is equal to the order of the characteristic equation.
4. *Symmetry of root loci*: The root loci are symmetrical about the real axis.
5. *Root locus locations on real axis*: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
6. *Breakaway points*: The points at which a locus breaks away from the real axis can be found by obtaining the roots of the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\{GH(z)\} = 0$$

7. *Unit circle crossover*: This can be obtained by determining the value of K for marginal stability using the Jury test, and substituting it in the characteristic equation (7.76).

Example 7.6 (See also Appendix 1, *examp76.m*)
Sketch the root locus diagram for Example 7.4, shown in Figure 7.14. Determine the breakaway points, the value of K for marginal stability and the unit circle crossover.