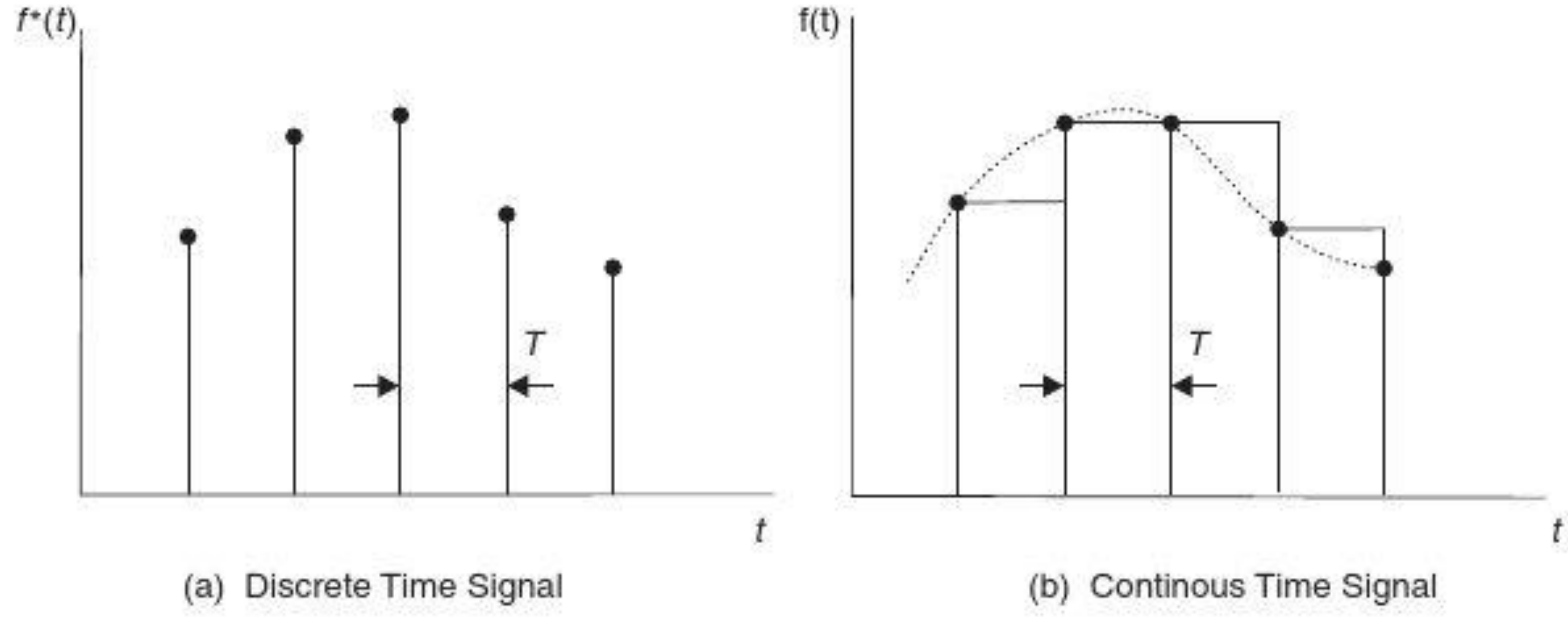

Fig. 7.6 Construction of a continuous signal using a zero-order hold.

7.4 The *z*-transform

The z-transform is the principal analytical tool for single-input–single-output discrete-time systems, and is analogous to the Laplace transform for continuous systems.

Conceptually, the symbol z can be associated with discrete time shifting in a difference equation in the same way that s can be associated with differentiation in a differential equation.

Taking Laplace transforms of equation (7.1), which is the ideal sampled signal, gives

$$F^*(s) = \mathscr{L}[f^*(t)] = \sum_{k=0}^{\infty} f(kT)\mathrm{e}^{-kTs} \tag{7.3}$$

or

$$F^*(s) = \sum_{k=0}^{\infty} f(kT)\left(\mathrm{e}^{sT}\right)^{-k} \tag{7.4}$$

Define z as

$$z = \mathrm{e}^{sT} \tag{7.5}$$

then

$$F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k} = Z[f(t)] \tag{7.6}$$

In 'long-hand' form equation (7.6) is written as

$$F(z) = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots + f(kT)z^{-k} \tag{7.7}$$

Example 7.1

Find the z-transform of the unit step function $f(t) = 1$.

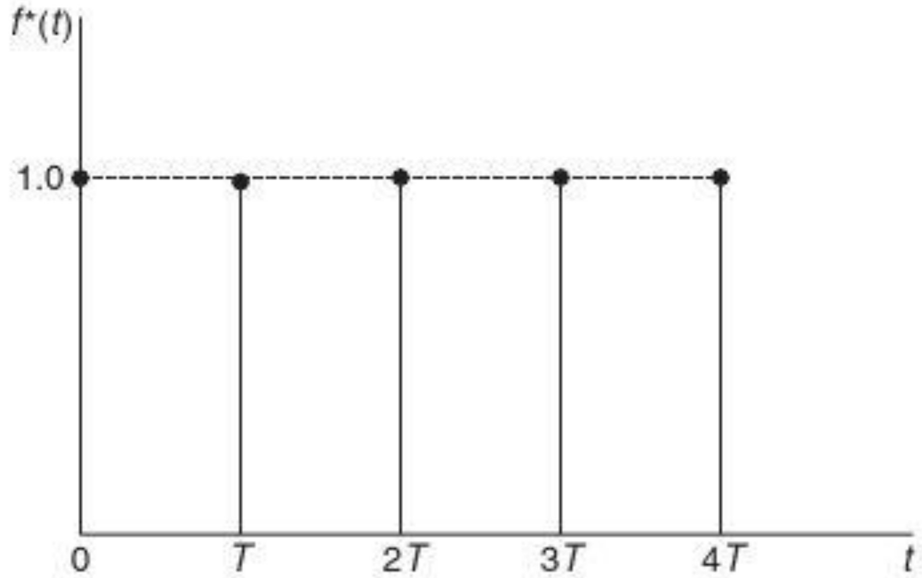

Fig. 7.7 z-Transform of a sampled unit step function.

Solution
From equations (7.6) and (7.7)

$$Z[1(t)] = \sum_{k=0}^{\infty} 1(kT)z^{-k} \tag{7.8}$$

or

$$F(z) = 1 + z^{-1} + z^{-2} + \ldots + z^{-k} \tag{7.9}$$

Figure 7.7 shows a graphical representation of equation (7.9).

Equation (7.9) can be written in 'closed form' as

$$Z[1(t)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \tag{7.10}$$

Equations (7.9) and (7.10) can be shown to be the same by long division

$$\begin{array}{r l} & 1 + z^{-1} + z^{-2} + \cdots \\ z-1 \big) & z \quad 0 \quad\quad 0 \\ & \underline{z-1} \\ & 0+1 \\ & \quad \underline{1-z^{-1}} \\ & \quad 0+z^{-1} \\ & \quad\quad z^{-1}-z^{-2} \end{array} \tag{7.11}$$

Table 7.1 gives Laplace and z-transforms of common functions.

z-transform Theorems:

(a) Linearity

$$Z[f_1(t) \pm f_2(t)] = F_1(z) \pm F_2(z) \tag{7.12}$$

Table 7.1 Common Laplace and z-transforms

	$f(t)$ or $f(kT)$	$F(s)$	$F(z)$
1	$\delta(t)$	1	1
2	$\delta(t-kT)$	e^{-kTs}	z^{-k}
3	$1(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$
4	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
5	e^{-at}	$\frac{1}{(s+a)}$	$\frac{z}{z-e^{-aT}}$
6	$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
7	$\frac{1}{a}(at-1+e^{-at})$	$\frac{a}{s^2(s+a)}$	$\frac{z\{(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})\}}{a(z-1)^2(z-e^{-aT})}$
8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
9	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
10	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
11	$e^{-at}\cos\omega t$	$\frac{(s+a)}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$

(b) Initial Value Theorem

$$f(0)=\lim_{z\to\infty} F(z) \tag{7.13}$$

(c) Final Value Theorem

$$f(\infty)=\lim_{z\to 1}\left[\left(\frac{z-1}{z}\right)F(z)\right] \tag{7.14}$$

7.4.1 Inverse transformation

The discrete time response can be found using a number of methods.

(a) Infinite power series method

Example 7.2

A sampled-data system has a transfer function

$$G(s)=\frac{1}{s+1}$$

If the sampling time is one second and the system is subject to a unit step input function, determine the discrete time response. (N.B. normally, a zero-order hold would be included, but, in the interest of simplicity, has been omitted.) Now

$$X_o(z) = G(z)X_i(z) \tag{7.15}$$

from Table 7.1

$$X_o(z) = \left(\frac{z}{z - e^{-T}}\right)\left(\frac{z}{z-1}\right) \tag{7.16}$$

for $T = 1$ second

$$\begin{aligned} X_o(z) &= \left(\frac{z}{z - 0.368}\right)\left(\frac{z}{z-1}\right) \\ &= \frac{z^2}{z^2 - 1.368z + 0.368} \end{aligned} \tag{7.17}$$

By long division

$$\begin{array}{r|l} & 1 + 1.368z^{-1} + 1.503z^{-2} + \cdots \\ \hline z^2 - 1.368z + 0.368 & z^2 \quad 0 \quad 0 \quad 0 \\ & \underline{z^2 - 1.368z + 0.368} \\ & 0 + 1.368z - 0.368 \\ & \quad \underline{1.368z - 1.871 + 0.503z^{-1}} \\ & \quad\quad 0 + 1.503 - 0.503z^{-1} \\ & \quad\quad\quad 1.503 - 2.056z^{-1} + 0.553z^{-2} \end{array} \tag{7.18}$$

Thus

$$x_o(0) = 1$$

$$x_o(1) = 1.368$$

$$x_o(2) = 1.503$$

(b) Difference equation method

Consider a system of the form

$$\frac{X_o}{X_i}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots} \tag{7.19}$$

Thus

$$(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots)X_o(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots)X_i(z) \tag{7.20}$$

or

$$X_o(z) = (-a_1 z^{-1} - a_2 z^{-2} - \cdots)X_o(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots)X_i(z) \tag{7.21}$$

Equation (7.21) can be expressed as a difference equation of the form

$$\begin{aligned} x_o(kT) = &-a_1 x_o(k-1)T - a_2 x_o(k-2)T - \cdots \\ &+ b_0 x_i(kT) + b_1 x_i(k-1)T + b_2 x_i(k-2)T + \cdots \end{aligned} \tag{7.22}$$

In Example 7.2

$$\frac{X_o}{X_i}(s) = \frac{1}{1+s}$$
$$= \frac{z}{z - e^{-T}} = \frac{z}{z - 0.368} \qquad (7.23)$$

Equation (7.23) can be written as

$$\frac{X_o}{X_i}(z) = \frac{1}{1 - 0.368z^{-1}} \qquad (7.24)$$

Equation (7.24) is in the same form as equation (7.19). Hence

$$(1 - 0.368z^{-1})X_o(z) = X_i(z)$$

or

$$X_o(z) = 0.368z^{-1}X_o(z) + X_i(z) \qquad (7.25)$$

Equation (7.25) can be expressed as a difference equation

$$x_o(kT) = 0.368x_o(k-1)T + x_i(kT) \qquad (7.26)$$

Assume that $x_o(-1) = 0$ and $x_i(kT) = 1$, then from equation (7.26)

$$x_o(0) = 0 + 1 = 1, \quad k = 0$$
$$x_o(1) = (0.368 \times 1) + 1 = 1.368, \quad k = 1$$
$$x_o(2) = (0.368 \times 1.368) + 1 = 1.503, \quad k = 2 \quad \text{etc.}$$

These results are the same as with the power series method, but difference equations are more suited to digital computation.

7.4.2 The pulse transfer function

Consider the block diagrams shown in Figure 7.8. In Figure 7.8(a) $U^*(s)$ is a sampled input to $G(s)$ which gives a continuous output $X_o(s)$, which when sampled by a

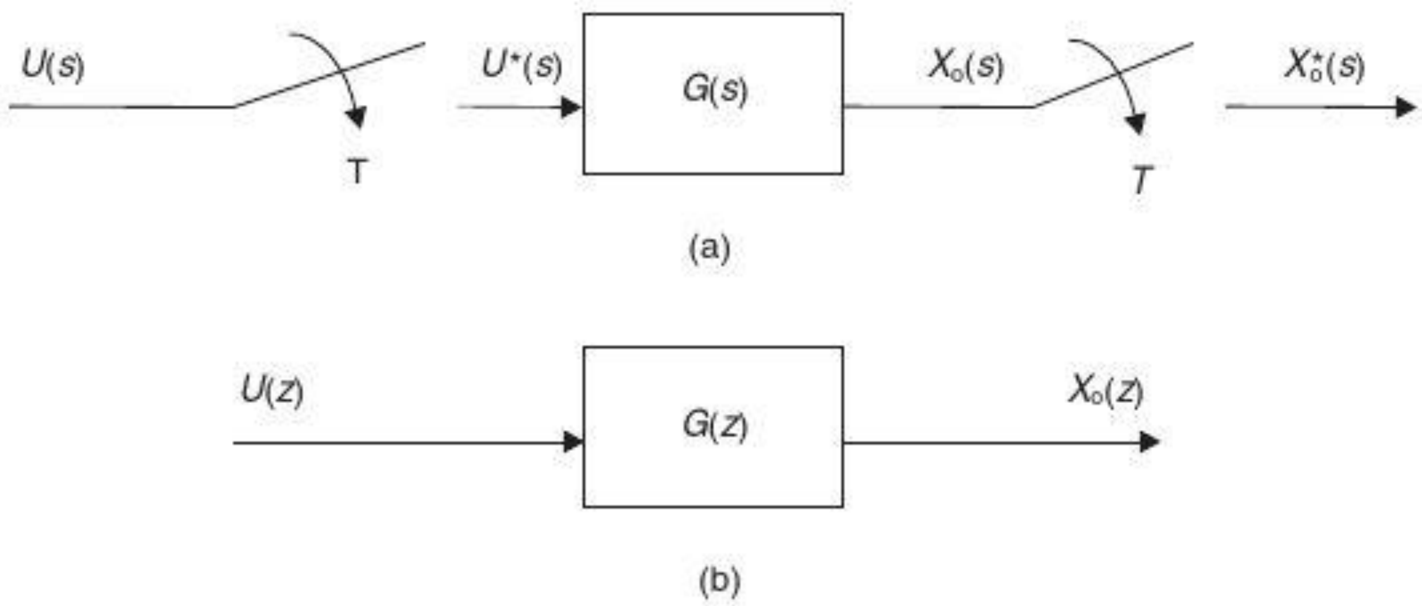

Fig. 7.8 Relationship between $G(s)$ and $G(z)$.