Since from equation (7.63) and Figure 7.16

$$z = |z| \angle \omega T \tag{7.89}$$

and $T = 0.5$, then the frequency of oscillation at the onset of instability is

$$\begin{aligned} 0.5\omega &= 1.33 \\ \omega &= 2.66\,\text{rad/s} \end{aligned} \tag{7.90}$$

The root locus diagram is shown in Figure 7.20.

It can be seen from Figure 7.20 that the complex loci form a circle. This is usually the case for second-order plant, where

$$\begin{aligned} \text{Radius} &= \sum |\text{open-loop poles}| \\ \text{Centre} &= (\text{Open-loop zero, } 0) \end{aligned} \tag{7.91}$$

The step response shown in Figure 7.15 is for $K = 1$. Inserting $K = 1$ into the characteristic equation gives

$$z^2 - 1.276z + 0.434 = 0$$

or

$$z = 0.638 \pm \text{j}0.164$$

This position is shown in Figure 7.20. The K values at the breakaway points are also shown in Figure 7.20.

7.7 Digital compensator design

In sections 5.4 and 6.6, compensator design in the s-plane and the frequency domain were discussed for continuous systems. In the same manner, digital compensators may be designed in the z-plane for discrete systems.

Figure 7.13 shows the general form of a digital control system. The pulse transfer function of the digital controller/compensator is written

$$\frac{U}{E}(z) = D(z) \tag{7.92}$$

and the closed-loop pulse transfer function become

$$\frac{C}{R}(z) = \frac{D(z)G(z)}{1 + D(z)GH(z)} \tag{7.93}$$

and hence the characteristic equation is

$$1 + D(z)GH(z) = 0 \tag{7.94}$$

7.7.1 Digital compensator types

In a continuous system, a differentiation of the error signal e can be represented as

$$u(t) = \frac{\mathrm{d}e}{\mathrm{d}t}$$

Taking Laplace transforms with zero initial conditions

$$\frac{U}{E}(s) = s \tag{7.95}$$

In a discrete system, a differentiation can be approximated to

$$u(kT) = \frac{e(kT) - e(k-1)T}{T}$$

hence

$$\frac{U}{E}(z) = \frac{1 - z^{-1}}{T} \tag{7.96}$$

Hence, the Laplace operator can be approximated to

$$s = \frac{1 - z^{-1}}{T} = \frac{z - 1}{Tz} \tag{7.97}$$

Digital PID controller: From equation (4.92), a continuous PID controller can be written as

$$\frac{U}{E}(s) = \frac{K_1(T_\mathrm{i}T_\mathrm{d}s^2 + T_\mathrm{i}s + 1)}{T_\mathrm{i}s} \tag{7.98}$$

Inserting equation (7.97) into (7.98) gives

$$\frac{U}{E}(z) = \frac{K_1\left\{T_\mathrm{i}T_\mathrm{d}\left(\frac{z-1}{Tz}\right)^2 + T_\mathrm{i}\left(\frac{z-1}{Tz}\right) + 1\right\}}{T_\mathrm{i}\left(\frac{z-1}{Tz}\right)} \tag{7.99}$$

which can be simplified to give

$$\frac{U}{E}(z) = \frac{K_1(b_2z^2 + b_1z + b_0)}{z(z-1)} \tag{7.100}$$

where

$$\begin{aligned} b_0 &= \frac{T_\mathrm{d}}{T} \\ b_1 &= \left(1 - \frac{2T_\mathrm{d}}{T}\right) \\ b_2 &= \left(\frac{T_\mathrm{d}}{T} + \frac{T}{T_\mathrm{i}} + 1\right) \end{aligned} \tag{7.101}$$

Tustin's Rule: Tustin's rule, also called the bilinear transformation, gives a better approximation to integration since it is based on a trapizoidal rather than a rectangular area. Tustin's rule approximates the Laplace transform to

$$s = \frac{2(z-1)}{T(z+1)} \tag{7.102}$$

Inserting this value of s into the denominator of equation (7.98), still yields a digital PID controller of the form shown in equation (7.100) where

$$\begin{aligned} b_0 &= \frac{T_\mathrm{d}}{T} \\ b_1 &= \left(\frac{T}{2T_\mathrm{i}} - \frac{2T_\mathrm{d}}{T} - 1\right) \\ b_2 &= \left(\frac{T}{2T_\mathrm{i}} + \frac{T_\mathrm{d}}{T} + 1\right) \end{aligned} \tag{7.103}$$

Example 7.7 (See also Appendix 1, *examp77.m*)
The laser guided missile shown in Figure 5.26 has an open-loop transfer function (combining the fin dynamics and missile dynamics) of

$$G(s)H(s) = \frac{20}{s^2(s+5)} \tag{7.104}$$

A lead compensator, see case study Example 6.6, and equation (6.113) has a transfer function of

$$G(s) = \frac{0.8(1+s)}{(1+0.0625s)} \tag{7.105}$$

(a) Find the z-transform of the missile by selecting a sampling frequency of at least 10 times higher than the system bandwidth.
(b) Convert the lead compensator in equation (7.105) into a digital compensator using the simple method, i.e. equation (7.97) and find the step response of the system.
(c) Convert the lead compensator in equation (7.105) into a digital compensator using Tustin's rule, i.e. equation (7.102) and find the step response of the system.
(d) Compare the responses found in (b) and (c) with the continuous step response, and convert the compensator that is closest to this into a difference equation.

Solution

(a) From Figure 6.39, lead compensator two, the bandwidth is 5.09 rad/s, or 0.81 Hz. Ten times this is 8.1 Hz, so select a sampling frequency of 10 Hz, i.e.

$T = 0.1$ seconds. For a sample and hold device cascaded with the missile dynamics

$$G(s) = \left(\frac{1 - e^{-Ts}}{s}\right)\left\{\frac{20}{s^2(s+5)}\right\} \tag{7.106}$$

$$G(s) = (1 - e^{-Ts})\left\{\frac{20}{s^3(s+5)}\right\} \tag{7.107}$$

For $T = 0.1$, equation (7.107) has a z-transform of

$$G(z) = \frac{0.00296z^2 + 0.01048z + 0.0023}{z^3 - 2.6065z^2 + 2.2131z - 0.6065} \tag{7.108}$$

(b) Substituting

$$s = \frac{z-1}{Tz}$$

into lead compensator given in equation (7.105) to obtain digital compensator

$$D(z) = 0.8\left\{\frac{\frac{Tz+(z-1)}{Tz}}{\frac{Tz+0.0625(z-1)}{Tz}}\right\}$$

This simplifies to give

$$D(z) = \frac{5.4152z - 4.923}{z - 0.3846} \tag{7.109}$$

(c) Using Tustin's rule

$$s = \frac{2(z-1)}{T(z+1)}$$

Substituting into lead compensator

$$D(z) = 0.8\left[\frac{\frac{T(z+1)+2(z-1)}{T(z+1)}}{\frac{T(z+1)+0.0625\{2(z-1)\}}{T(z+1)}}\right]$$

This simplifies to give

$$D(z) = \frac{7.467z - 6.756}{z - 0.111} \tag{7.110}$$

(d) From Figure 7.21, it can be seen that the digital compensator formed using Tustin's rule is closest to the continuous response. From equation (7.110)

$$\frac{U}{E}(z) = \frac{7.467 - 6.756z^{-1}}{1 - 0.111z^{-1}} \tag{7.111}$$

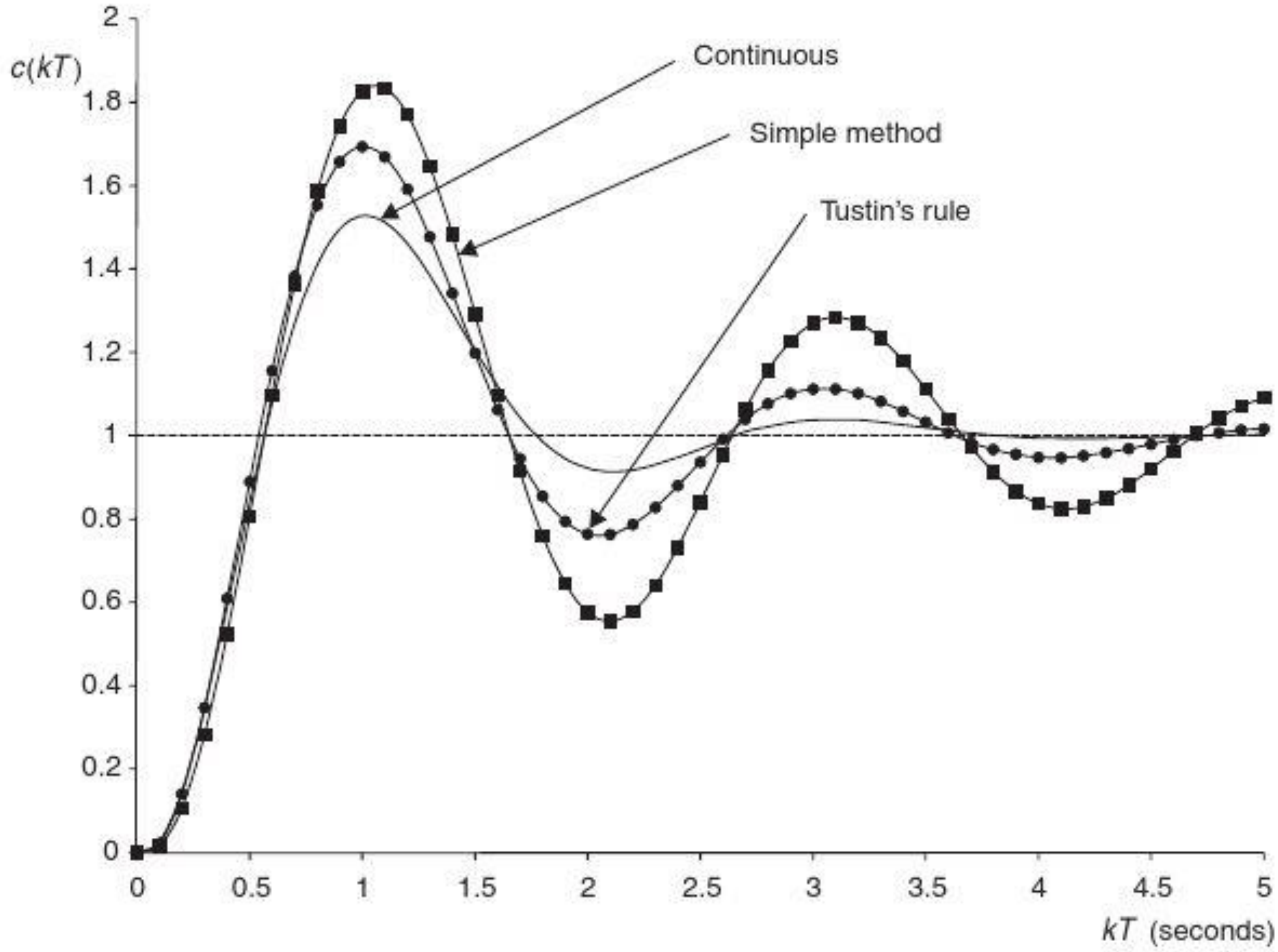

Fig. 7.21 Comparison between discrete and continuous response.

Hence the difference equation for the digital compensator is

$$u(kT) = 0.111u(k-1)T + 7.467e(kT) - 6.756e(k-1)T \tag{7.112}$$

7.7.2 Digital compensator design using pole placement

Case study

Example 7.8 (See also Appendix 1, *examp78.m*)
The continuous control system shown in Figure 7.22(a) is to be replaced by the digital control system shown in Figure 7.22(b).

(a) For the continuous system, find the value of K that gives the system a damping ratio of 0.5. Determine the closed-loop poles in the s-plane and hence the values of σ and ω.
(b) Find the closed-loop bandwidth ω_b and make the sampling frequency ω_s a factor of 10 higher. What is the value of T?
(c) For the sampled system shown in Figure 7.22(b), find the open-loop pulse transfer function $G(z)$ when the sample and hold device is in cascade with the plant.
(d) With $D(z)$ set to the value of K found in (a), compare the continuous and discrete step responses.