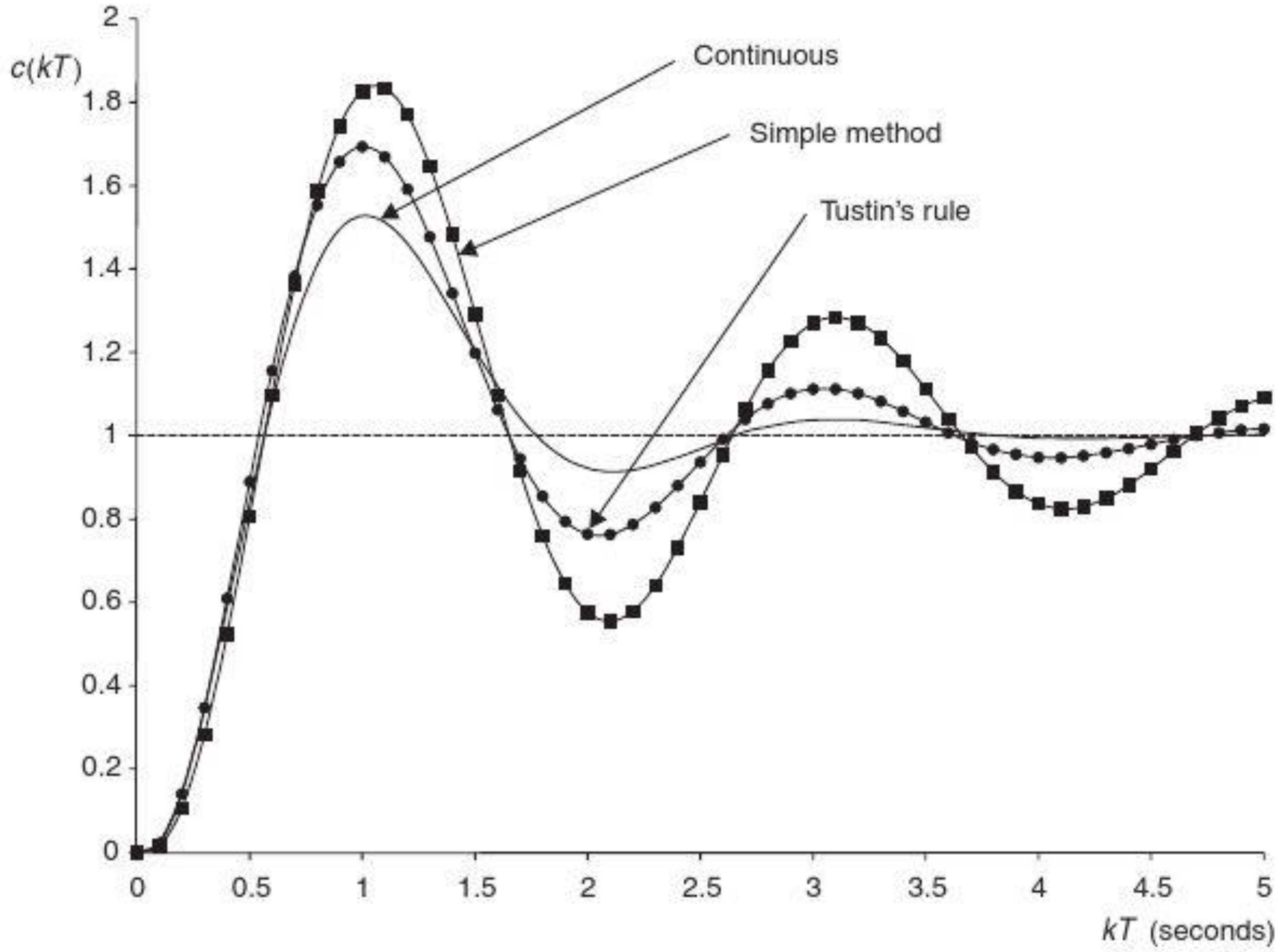

Fig. 7.21 Comparison between discrete and continuous response.

Hence the difference equation for the digital compensator is

$$u(kT) = 0.111u(k-1)T + 7.467e(kT) - 6.756e(k-1)T \tag{7.112}$$

7.7.2 Digital compensator design using pole placement

Case study

Example 7.8 (See also Appendix 1, *examp78.m*)
The continuous control system shown in Figure 7.22(a) is to be replaced by the digital control system shown in Figure 7.22(b).

(a) For the continuous system, find the value of K that gives the system a damping ratio of 0.5. Determine the closed-loop poles in the s-plane and hence the values of σ and ω.
(b) Find the closed-loop bandwidth ω_b and make the sampling frequency ω_s a factor of 10 higher. What is the value of T?
(c) For the sampled system shown in Figure 7.22(b), find the open-loop pulse transfer function $G(z)$ when the sample and hold device is in cascade with the plant.
(d) With $D(z)$ set to the value of K found in (a), compare the continuous and discrete step responses.

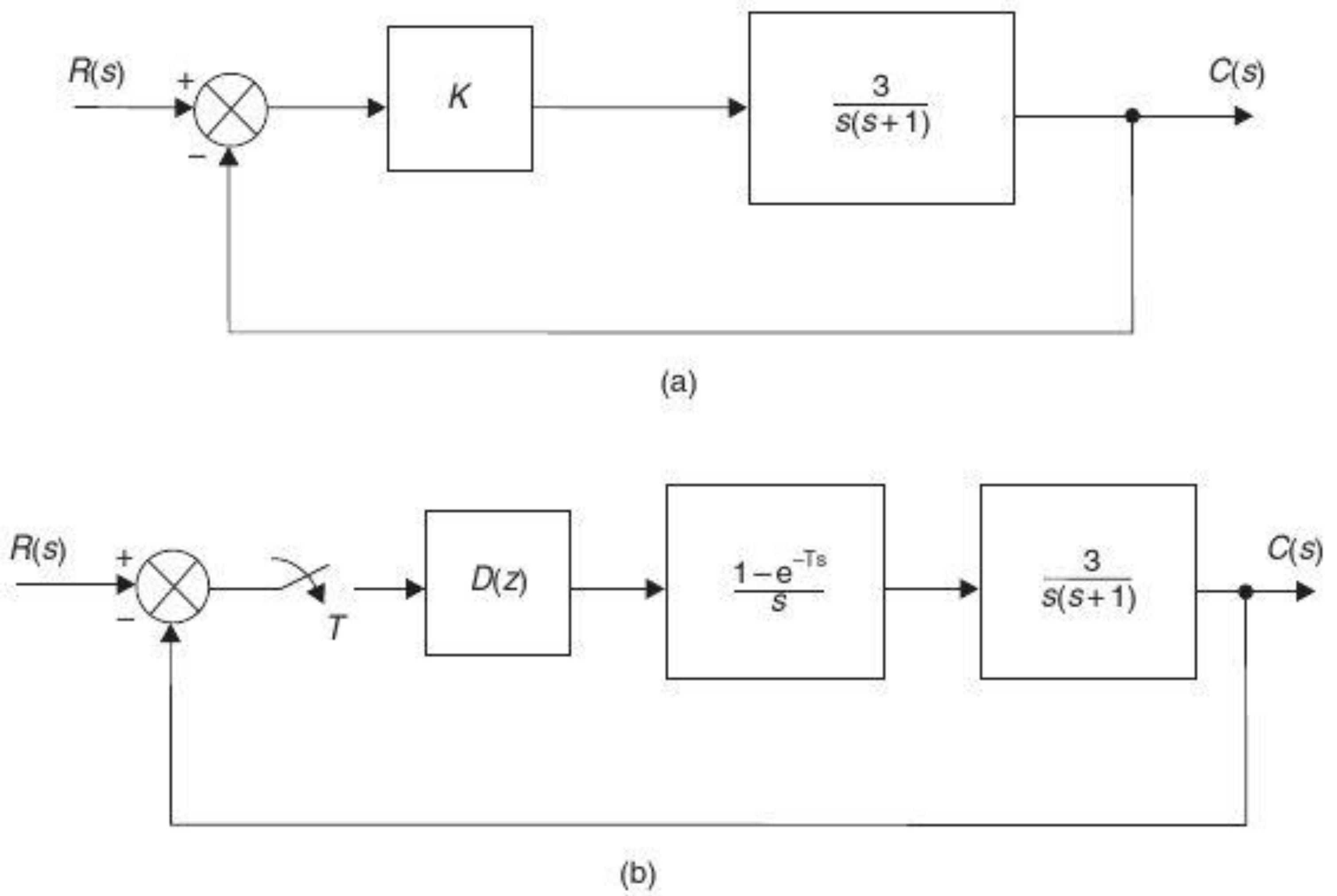

Fig. 7.22 Continuous and digital control systems.

(e) By mapping the closed-loop poles from the s to the z-plane, design a compensator $D(z)$ such that both continuous and sampled system have identical closed-loop response, i.e. $\zeta = 0.5$.

Solution

(a) The root-locus diagram for the continuous system in shown in Figure 7.23. From Figure 7.23 the closed-loop poles are

$$s = -0.5 \pm \mathrm{j}0.866 \tag{7.113}$$

or

$$\sigma = -0.5, \quad \omega = 0.866\,\mathrm{rad/s}$$

and the value of K is 0.336.

(b) Plotting the closed-loop frequency response for the continuous system gives a bandwidth ω_b of 1.29 rad/s(0.205 Hz). The sampling frequency should therefore be a factor of 10 higher, i.e. 12.9 rad/s(2.05 Hz). Rounding down to 2.0 Hz gives a sampling time T of 0.5 seconds.

(c)
$$G(z) = (1 - z^{-1})Z\left\{\frac{3}{s^2(s+1)}\right\} \tag{7.114}$$

Using transform 7 in Table 7.1

$$G(z) = \frac{3\{(\mathrm{e}^{-0.5} - 0.5)z + (1 - 1.5\mathrm{e}^{-0.5})\}}{(z-1)(z-\mathrm{e}^{-0.5})}$$

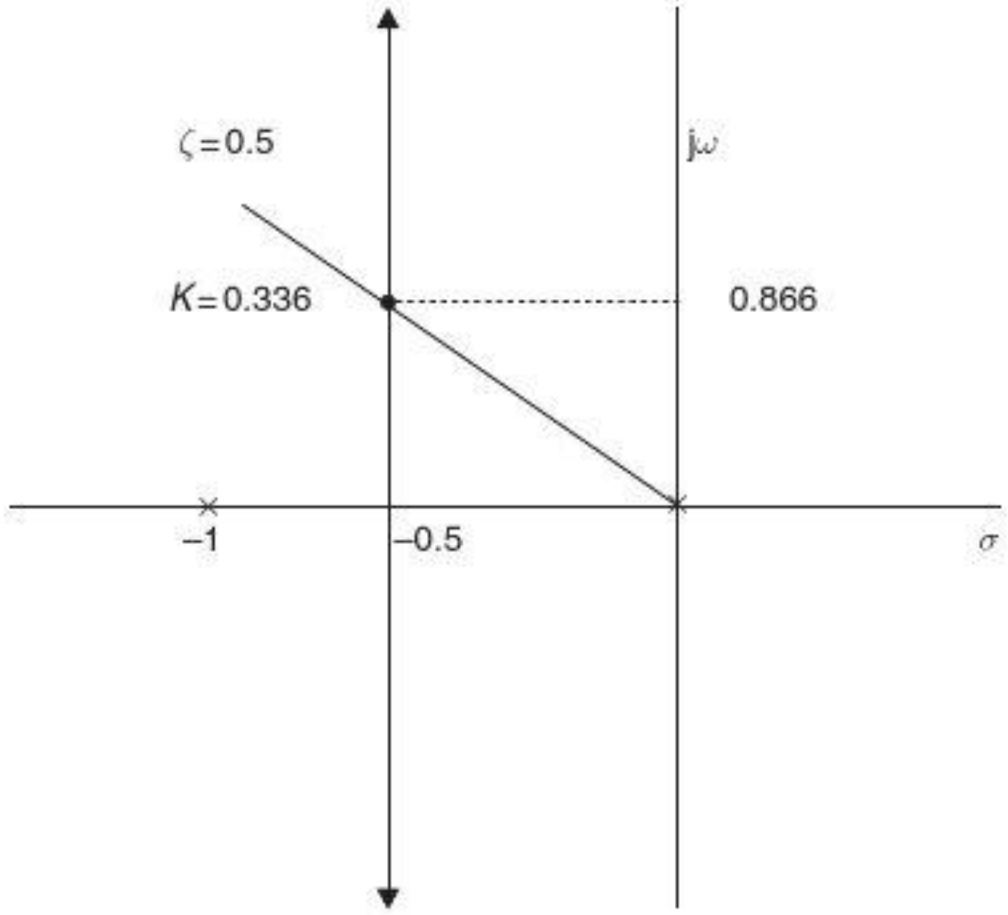

Fig. 7.23 Root locus diagram for continuous system.

Hence

$$G(z) = \frac{0.3196(z + 0.8467)}{(z - 1)(z - 0.6065)} \tag{7.115}$$

(d) With $D(z) = K = 0.336$, the difference between the continuous and discrete step response can be seen in Figure 7.24.

(e) Mapping closed-loop poles from s to z-plane

$$|z| = e^{\sigma T}$$

inserting values

$$|z| = e^{-0.5 \times 0.5} = 0.779 \tag{7.116}$$

$$\begin{aligned} \angle z &= \omega T \\ &= 0.866 \times 0.5 = 0.433 \text{ rad} \\ &= 24.8^\circ \end{aligned} \tag{7.117}$$

Converting from polar to cartesian co-ordinates gives the closed-loop poles in the z-plane

$$z = 0.707 \pm j0.327 \tag{7.118}$$

which provides a z-plane characteristic equation

$$z^2 - 1.414z + 0.607 = 0 \tag{7.119}$$

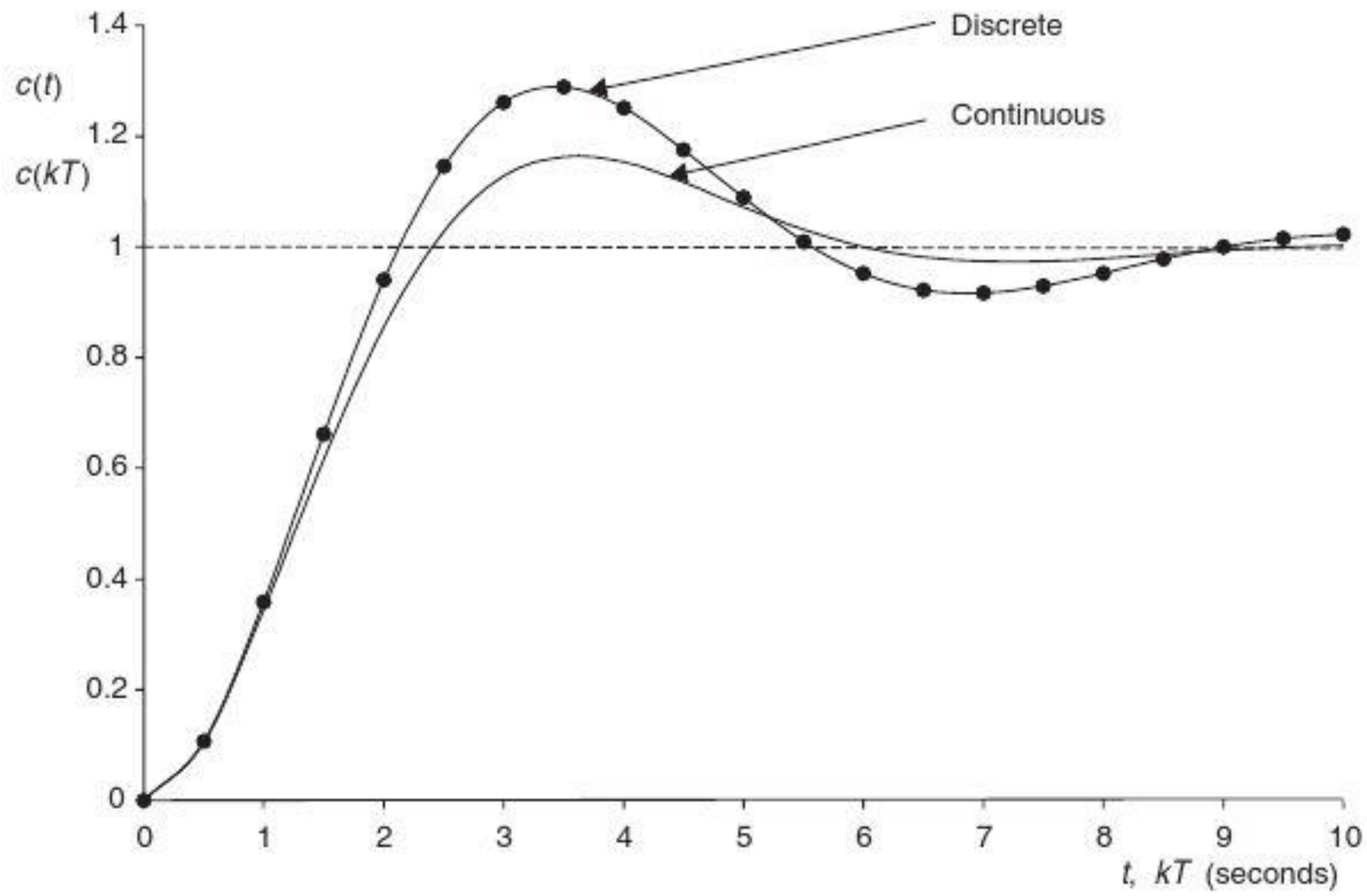

Fig. 7.24 Continuous and digital controllers set to $K = 0.336$.

The control problem is to design a compensator $D(z)$, which, when cascaded with $G(z)$, provides a characteristic equation

$$1 + D(z)G(z) = 0 \tag{7.120}$$

such that the equations (7.119) and (7.120) are identical. Let the compensator be of the form

$$D(z) = \frac{K(z - a)}{(z + b)} \tag{7.121}$$

Select the value of a so that the non-unity pole in $G(z)$ is cancelled

$$D(z)G(z) = \frac{K(z - 0.6065)}{(z + b)} \cdot \frac{0.3196(z + 0.8467)}{(z - 1)(z - 0.6065)} \tag{7.122}$$

Hence the characteristic equation (7.120) becomes

$$1 + \frac{0.3196K(z + 0.8467)}{(z + b)(z - 1)} = 0$$

which simplifies to give

$$z^2 + (0.3196K + b - 1)z + (0.2706K - b) = 0 \tag{7.123}$$

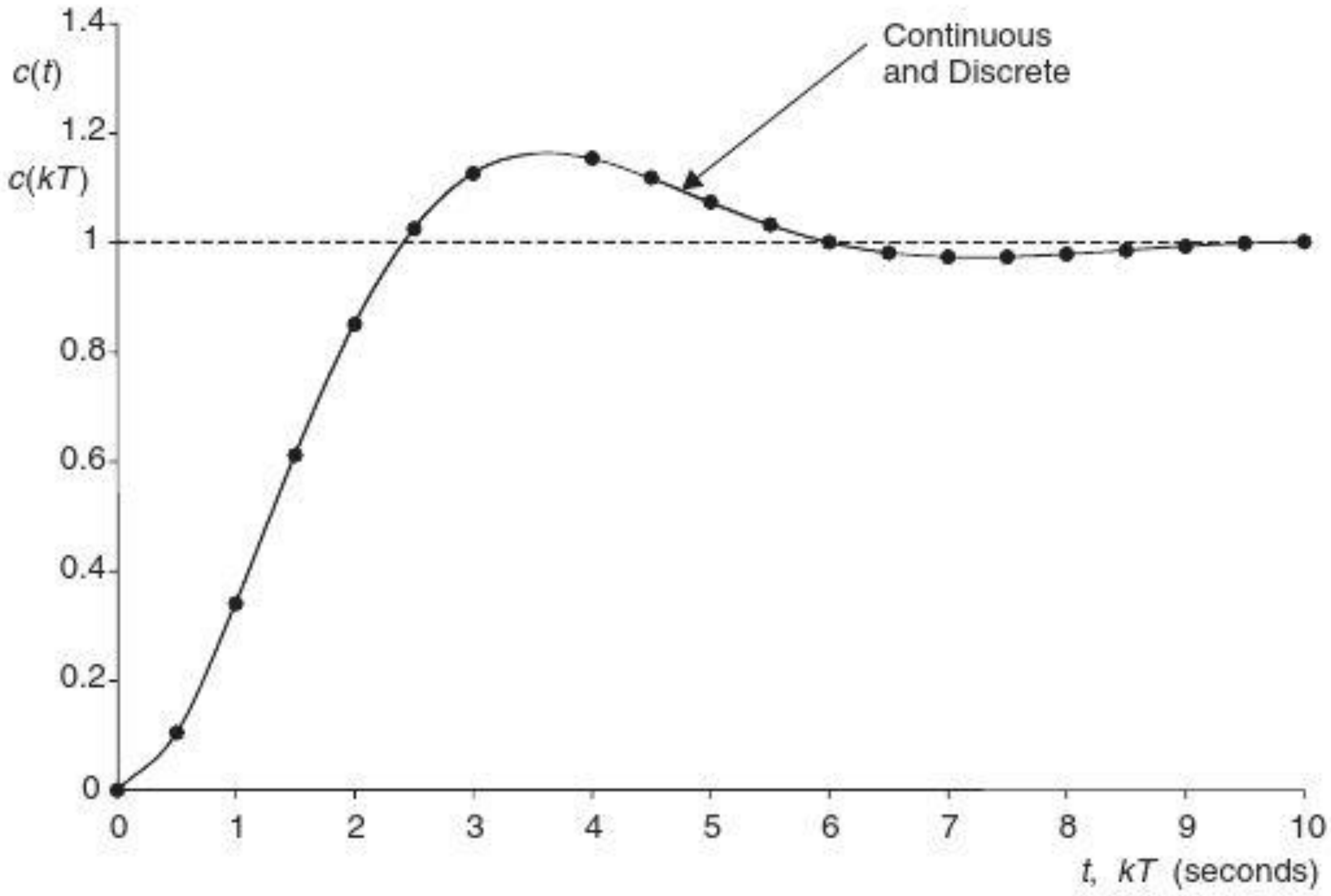

Fig. 7.25 Identical continuous and discrete step responses as a result of pole placement.

Equating coefficients in equations (7.119) and (7.123) gives

$$0.3196K + b - 1 = -1.414 \tag{7.124}$$

$$0.2706K - b \quad = 0.607 \tag{7.125}$$

$$\text{Add} \quad 0.5902K - 1 \quad = -0.807$$

or

$$\begin{aligned} 0.5902K &= 0.193 \\ K &= 0.327 \end{aligned} \tag{7.126}$$

Inserting equation (7.126) into (7.125)

$$\begin{aligned} (0.2706 \times 0.327) - 0.607 &= b \\ b &= -0.519 \end{aligned} \tag{7.127}$$

Thus the required compensator is

$$D(z) = \frac{U}{E}(z) = \frac{0.327(z - 0.6065)}{(z - 0.519)} \tag{7.128}$$

Figure 7.25 shows that the continuous and discrete responses are identical, both with $\zeta = 0.5$. The control algorithm can be implemented as a difference equation

$$\frac{U}{E}(z) = 0.327\frac{(1 - 0.6065z^{-1})}{(1 - 0.519z^{-1})} \tag{7.129}$$

hence

$$u(kT) = 0.327\mathrm{e}(kT) - 0.1983\mathrm{e}(k-1)T + 0.519u(k-1)T \tag{7.130}$$