In Example 7.2

$$\frac{X_o}{X_i}(s) = \frac{1}{1+s}$$
$$= \frac{z}{z - e^{-T}} = \frac{z}{z - 0.368} \qquad (7.23)$$

Equation (7.23) can be written as

$$\frac{X_o}{X_i}(z) = \frac{1}{1 - 0.368z^{-1}} \qquad (7.24)$$

Equation (7.24) is in the same form as equation (7.19). Hence

$$(1 - 0.368z^{-1})X_o(z) = X_i(z)$$

or

$$X_o(z) = 0.368z^{-1}X_o(z) + X_i(z) \qquad (7.25)$$

Equation (7.25) can be expressed as a difference equation

$$x_o(kT) = 0.368x_o(k-1)T + x_i(kT) \qquad (7.26)$$

Assume that $x_o(-1) = 0$ and $x_i(kT) = 1$, then from equation (7.26)

$$x_o(0) = 0 + 1 = 1, \quad k = 0$$
$$x_o(1) = (0.368 \times 1) + 1 = 1.368, \quad k = 1$$
$$x_o(2) = (0.368 \times 1.368) + 1 = 1.503, \quad k = 2 \quad \text{etc.}$$

These results are the same as with the power series method, but difference equations are more suited to digital computation.

7.4.2 The pulse transfer function

Consider the block diagrams shown in Figure 7.8. In Figure 7.8(a) $U^*(s)$ is a sampled input to $G(s)$ which gives a continuous output $X_o(s)$, which when sampled by a

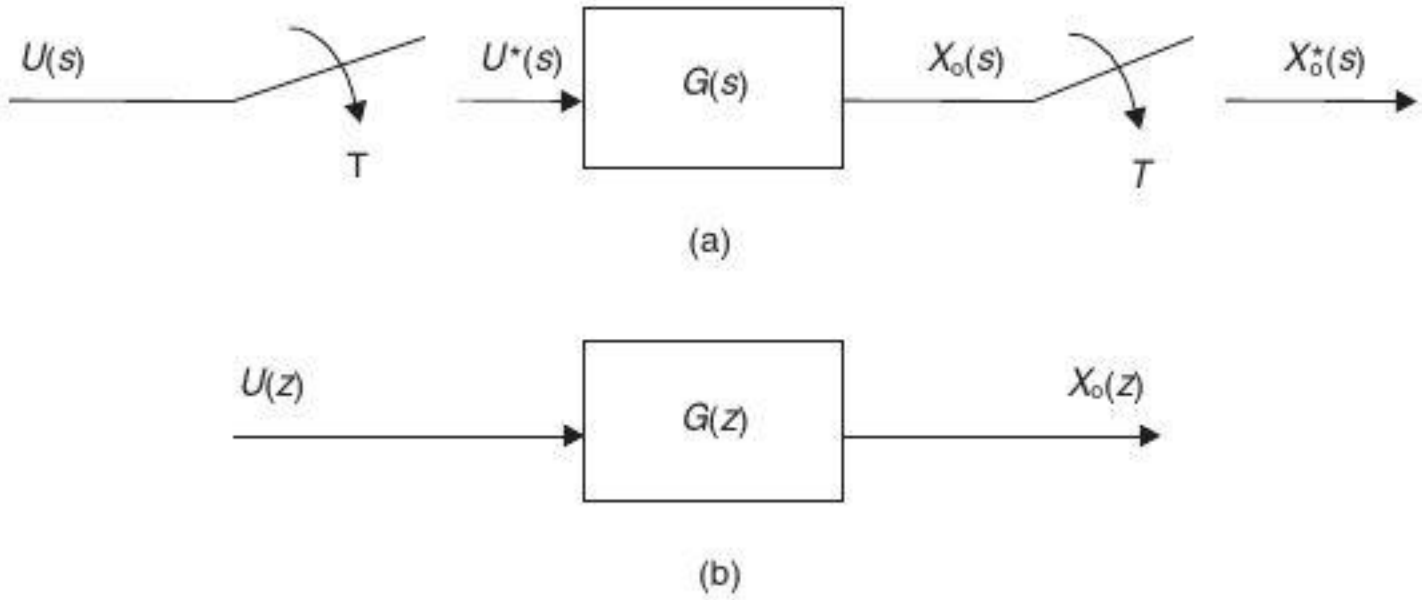

Fig. 7.8 Relationship between $G(s)$ and $G(z)$.

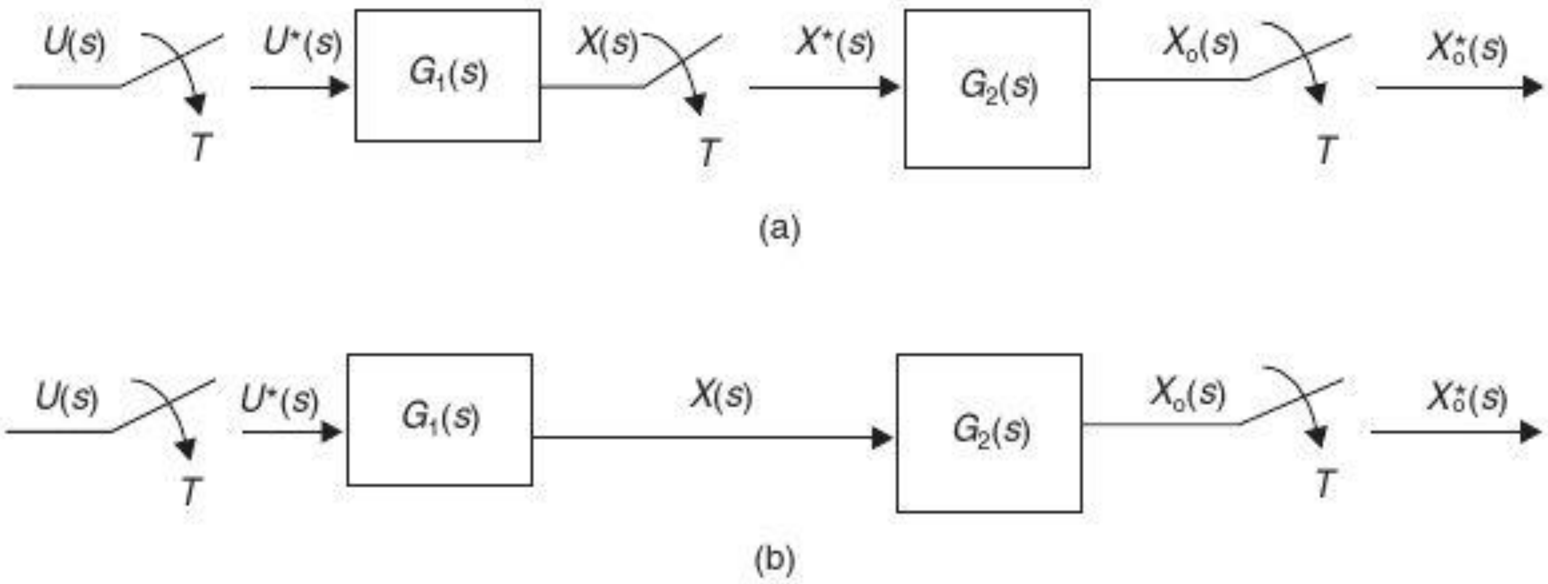

Fig. 7.9 Blocks in cascade.

synchronized sampler becomes $X_o^*(s)$. Figure 7.8(b) shows the pulse transfer function where $U(z)$ is equivalent to $U^*(s)$ and $X_o(z)$ is equivalent to $X_o^*(s)$.

From Figure 7.8(b) the pulse transfer function is

$$\frac{X_o}{U}(z) = G(z) \tag{7.27}$$

Blocks in Cascade: In Figure 7.9(a) there are synchronized samplers either side of blocks $G_1(s)$ and $G_2(s)$. The pulse transfer function is therefore

$$\frac{X_o}{U}(z) = G_1(z)G_2(z) \tag{7.28}$$

In Figure 7.9(b) there is no sampler between $G_1(s)$ and $G_2(s)$ so they can be combined to give $G_1(s)G_2(s)$, or $G_1G_2(s)$. Hence the output $X_o(z)$ is given by

$$X_o(z) = Z\{G_1G_2(s)\}U(z) \tag{7.29}$$

and the pulse transfer function is

$$\frac{X_o}{U}(z) = G_1G_2(z) \tag{7.30}$$

Note that $G_1(z)G_2(z) \neq G_1G_2(z)$.

Example 7.3 (See also Appendix 1, *examp73.m*)
A first-order sampled-data system is shown in Figure 7.10.
Find the pulse transfer function and hence calculate the response to a unit step and unit ramp. $T = 0.5$ seconds. Compare the results with the continuous system response $x_o(t)$. The system is of the type shown in Figure 7.9(b) and therefore

$$G(s) = G_1G_2(s)$$

Inserting values

$$G(s) = (1 - e^{-Ts})\left\{\frac{1}{s(s+1)}\right\} \tag{7.31}$$

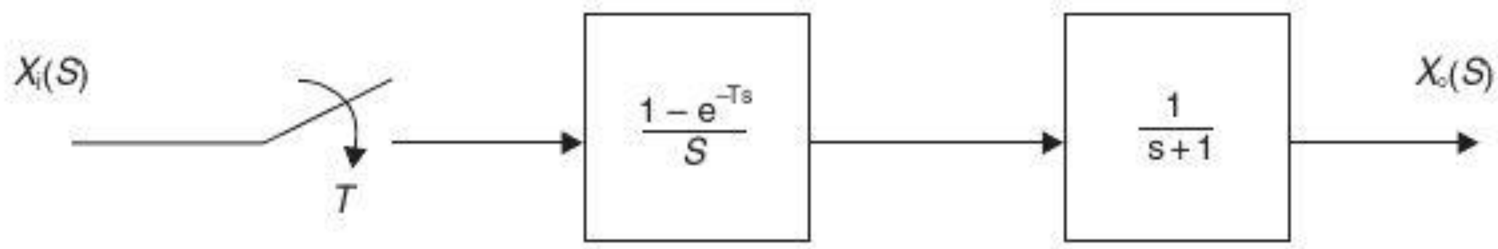

Fig. 7.10 First-order sampled-data system.

Taking z-transforms using Table 7.1.

$$G(z) = (1 - z^{-1})\left\{\frac{z(1 - e^{-T})}{(z-1)(z - e^{-T})}\right\} \tag{7.32}$$

or

$$G(z) = \left(\frac{z-1}{z}\right)\left\{\frac{z(1 - e^{-T})}{(z-1)(z - e^{-T})}\right\} \tag{7.33}$$

which gives

$$G(z) = \left(\frac{1 - e^{-T}}{z - e^{-T}}\right) \tag{7.34}$$

For $T = 0.5$ seconds

$$G(z) = \left(\frac{0.393}{z - 0.607}\right) \tag{7.35}$$

hence

$$\frac{X_o}{X_i}(z) = \left(\frac{0.393z^{-1}}{1 - 0.607z^{-1}}\right) \tag{7.36}$$

which is converted into a difference equation

$$x_o(kT) = 0.607x_o(k-1)T + 0.393x_i(k-1)T \tag{7.37}$$

Table 7.2 shows the discrete response $x_o(kT)$ to a unit step function and is compared with the continuous response (equation 3.29) where

$$x_o(t) = (1 - e^{-t}) \tag{7.38}$$

From Table 7.2, it can be seen that the discrete and continuous step response is identical. Table 7.3 shows the discrete response $x(kT)$ and continuous response $x(t)$ to a unit ramp function where $x_o(t)$ is calculated from equation (3.39)

$$x_o(t) = t - 1 + e^{-t} \tag{7.39}$$

In Table 7.3 the difference between $x_o(kT)$ and $x_o(t)$ is due to the sample and hold. It should also be noted that with the discrete response $x(kT)$, there is only knowledge of the output at the sampling instant.

Table 7.2 Comparison between discrete and continuous step response

k	kT (seconds)	$x_i(kT)$	$x_o(kT)$	$x_o(t)$
−1	−0.5	0	0	0
0	0	1	0	0
1	0.5	1	0.393	0.393
2	1.0	1	0.632	0.632
3	1.5	1	0.776	0.776
4	2.0	1	0.864	0.864
5	2.5	1	0.918	0.918
6	3.0	1	0.950	0.950
7	3.5	1	0.970	0.970
8	4.0	1	0.982	0.982

Table 7.3 Comparison between discrete and continuous ramp response

k	kT (seconds)	$x_i(kT)$	$x_o(kT)$	$x_o(t)$
−1	−0.5	0	0	0
0	0	0	0	0
1	0.5	0.5	0	0.107
2	1.0	1.0	0.304	0.368
3	1.5	1.5	0.577	0.723
4	2.0	2.0	0.940	1.135
5	2.5	2.5	1.357	1.582
6	3.0	3.0	1.805	2.050
7	3.5	3.5	2.275	2.530
8	4.0	4.0	2.757	3.018

7.4.3 The closed-loop pulse transfer function

Consider the error sampled system shown in Figure 7.11. Since there is no sampler between $G(s)$ and $H(s)$ in the closed-loop system shown in Figure 7.11, it is a similar arrangement to that shown in Figure 7.9(b). From equation (4.4), the closed-loop pulse transfer function can be written as

$$\frac{C}{R}(z) = \frac{G(z)}{1 + GH(z)} \tag{7.40}$$

In equation (7.40)

$$GH(z) = Z\{GH(s)\} \tag{7.41}$$

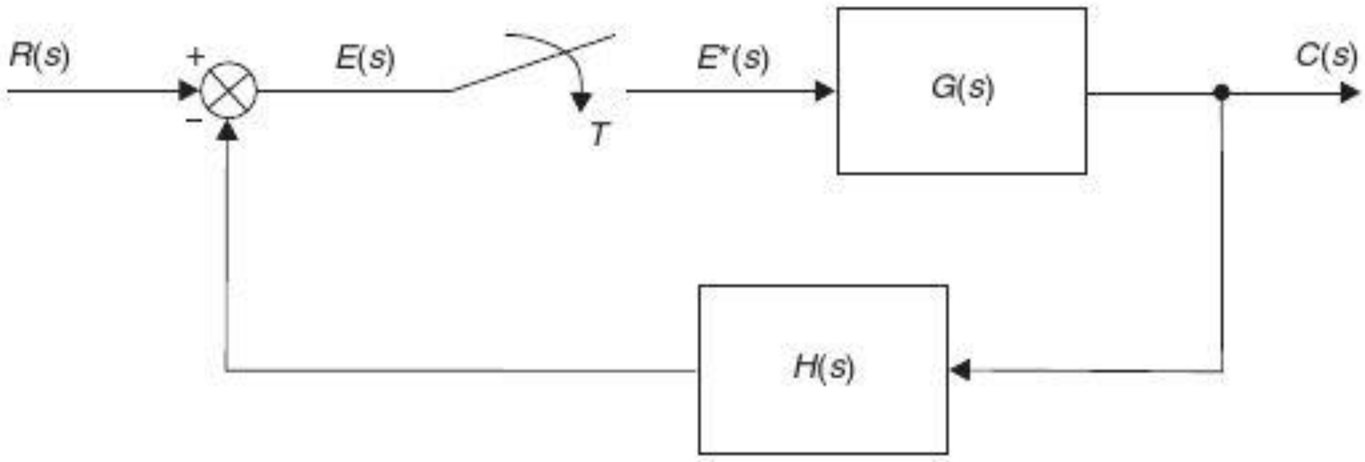

Fig. 7.11 Closed-loop error sampled system.

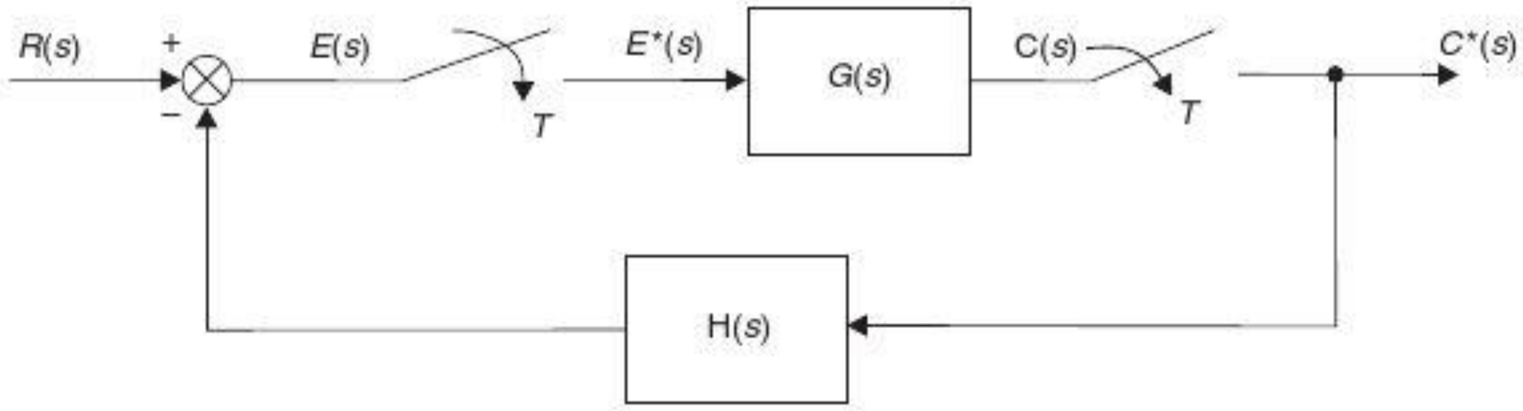

Fig. 7.12 Closed-loop error and output sampled system.

Consider the error and output sampled system shown in Figure 7.12. In Figure 7.12, there is now a sampler between $G(s)$ and $H(s)$, which is similar to Figure 7.9(a). From equation (4.4), the closed-loop pulse transfer function is now written as

$$\frac{C}{R}(z) = \frac{G(z)}{1 + G(z)H(z)} \tag{7.42}$$

7.5 Digital control systems

From Figure 7.1, a digital control system may be represented by the block diagram shown in Figure 7.13.

Example 7.4 (See also Appendix 1, *examp74.m*)
Figure 7.14 shows a digital control system. When the controller gain K is unity and the sampling time is 0.5 seconds, determine

(a) the open-loop pulse transfer function
(b) the closed-loop pulse transfer function
(c) the difference equation for the discrete time response
(d) a sketch of the unit step response assuming zero initial conditions
(e) the steady-state value of the system output

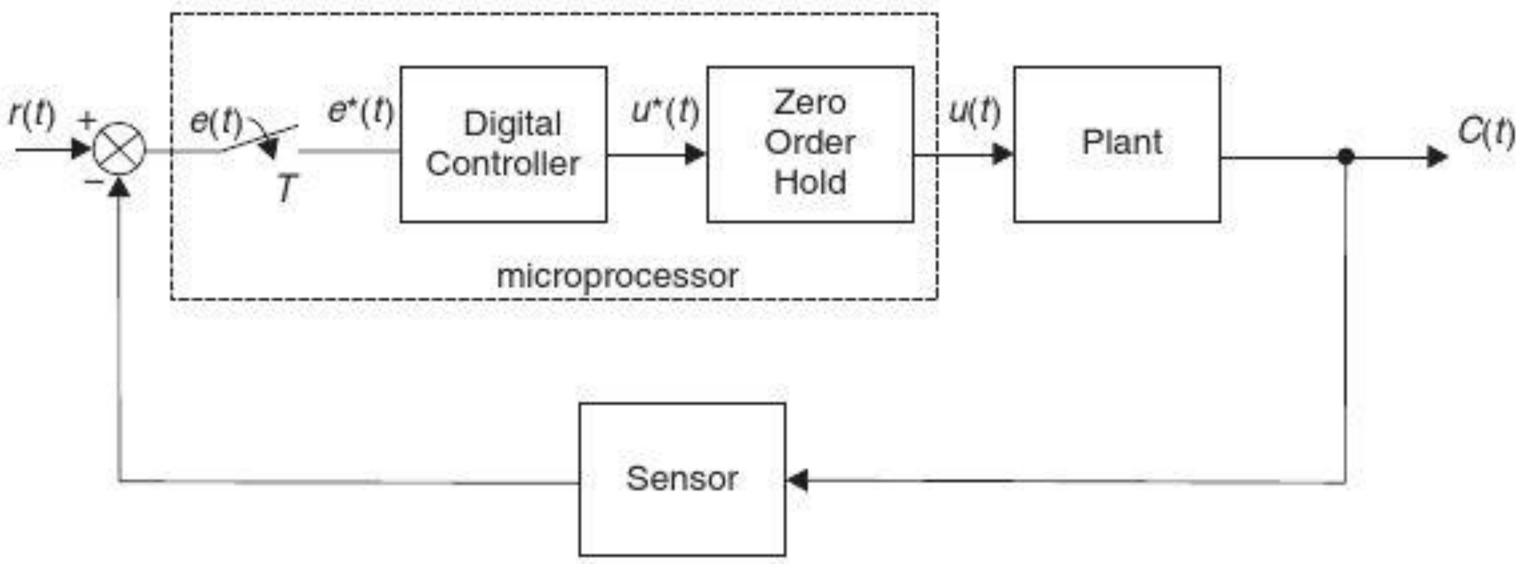

Fig. 7.13 Digital control system.

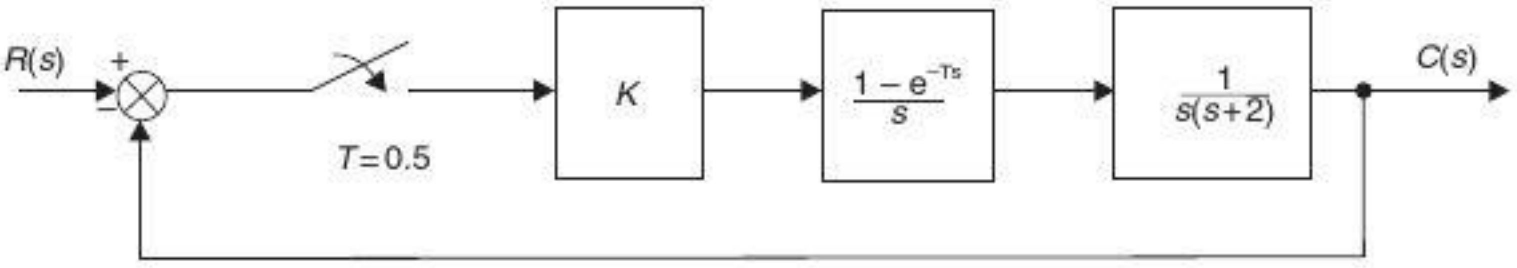

Fig. 7.14 Digital control system for Example 7.4.

Solution

(a) $$G(s) = K\left(\frac{1-e^{-Ts}}{s}\right)\left\{\frac{1}{s(s+2)}\right\} \tag{7.43}$$

Given $K = 1$

$$G(s) = \left(1-e^{-Ts}\right)\left\{\frac{1}{s^2(s+2)}\right\} \tag{7.44}$$

Partial fraction expansion

$$\frac{1}{s^2(s+2)} = \left\{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+2)}\right\} \tag{7.45}$$

or

$$1 = s(s+2)A + (s+2)B + s^2\mathrm{C} \tag{7.46}$$

Equating coefficients gives

$$A = -0.25$$
$$B = 0.5$$
$$C = 0.25$$

Substituting these values into equation (7.44) and (7.45)

$$G(s) = \left(1-e^{-Ts}\right)\left\{\frac{-0.25}{s} + \frac{0.5}{s^2} + \frac{0.25}{(s+2)}\right\} \tag{7.47}$$

or

$$G(s) = 0.25\left(1-e^{-Ts}\right)\left\{-\frac{1}{s} + \frac{2}{s^2} + \frac{1}{(s+2)}\right\} \tag{7.48}$$

Taking z-transforms

$$G(z) = 0.25\left(1-z^{-1}\right)\left\{\frac{-z}{(z-1)} + \frac{2Tz}{(z-1)^2} + \frac{z}{(z-e^{-2T})}\right\} \tag{7.49}$$

Given $T = 0.5$ seconds

$$G(z) = 0.25\left(\frac{z-1}{z}\right)z\left\{\frac{-1}{(z-1)} + \frac{2\times 0.5}{(z-1)^2} + \frac{1}{(z-0.368)}\right\} \tag{7.50}$$

Hence

$$G(z) = 0.25(z-1)\left\{\frac{-1(z-1)(z-0.368) + (z-0.368) + (z-1)^2}{(z-1)^2(z-0.368)}\right\} \tag{7.51}$$

$$G(z) = 0.25\left\{\frac{-z^2 + 1.368z - 0.368 + z - 0.368 + z^2 - 2z + 1}{(z-1)(z-0.368)}\right\} \tag{7.52}$$

which simplifies to give the open-loop pulse transfer function

$$G(z) = \left(\frac{0.092z + 0.066}{z^2 - 1.368z + 0.368}\right) \tag{7.53}$$

Note: This result could also have been obtained at equation (7.44) by using z-transform number 7 in Table 7.1, but the solution demonstrates the use of partial fractions.

(b) The closed-loop pulse transfer function, from equation (7.40) is

$$\frac{C}{R}(z) = \frac{\left(\frac{0.092z+0.066}{z^2-1.368z+0.368}\right)}{\left(1 + \frac{0.092z+0.066}{z^2-1.368z+0.368}\right)} \tag{7.54}$$

which simplifies to give the closed-loop pulse transfer function

$$\frac{C}{R}(z) = \frac{0.092z + 0.066}{z^2 - 1.276z + 0.434} \tag{7.55}$$

or

$$\frac{C}{R}(z) = \frac{0.092z^{-1} + 0.066z^{-2}}{1 - 1.276z^{-1} + 0.434z^{-2}} \tag{7.56}$$

(c) Equation (7.56) can be expressed as a difference equation

$$c(kT) = 1.276c(k-1)T - 0.434c(k-2)T + 0.092r(k-1)T + 0.066r(k-2)T \tag{7.57}$$

(d) Using the difference equation (7.57), and assuming zero initial conditions, the unit step response is shown in Figure 7.15.

Note that the response in Figure 7.15 is constructed solely from the knowledge of the two previous sampled outputs and the two previous sampled inputs.

(e) Using the final value theorem given in equation (7.14)

$$c(\infty) = \lim_{z\to 1}\left[\left(\frac{z-1}{z}\right)\frac{C}{R}(z)R(z)\right] \tag{7.58}$$

$$c(\infty) = \lim_{z\to 1}\left[\left(\frac{z-1}{z}\right)\left\{\frac{0.092z + 0.066}{1 - 1.276z + 0.434}\right\}\left(\frac{z}{z-1}\right)\right] \tag{7.59}$$

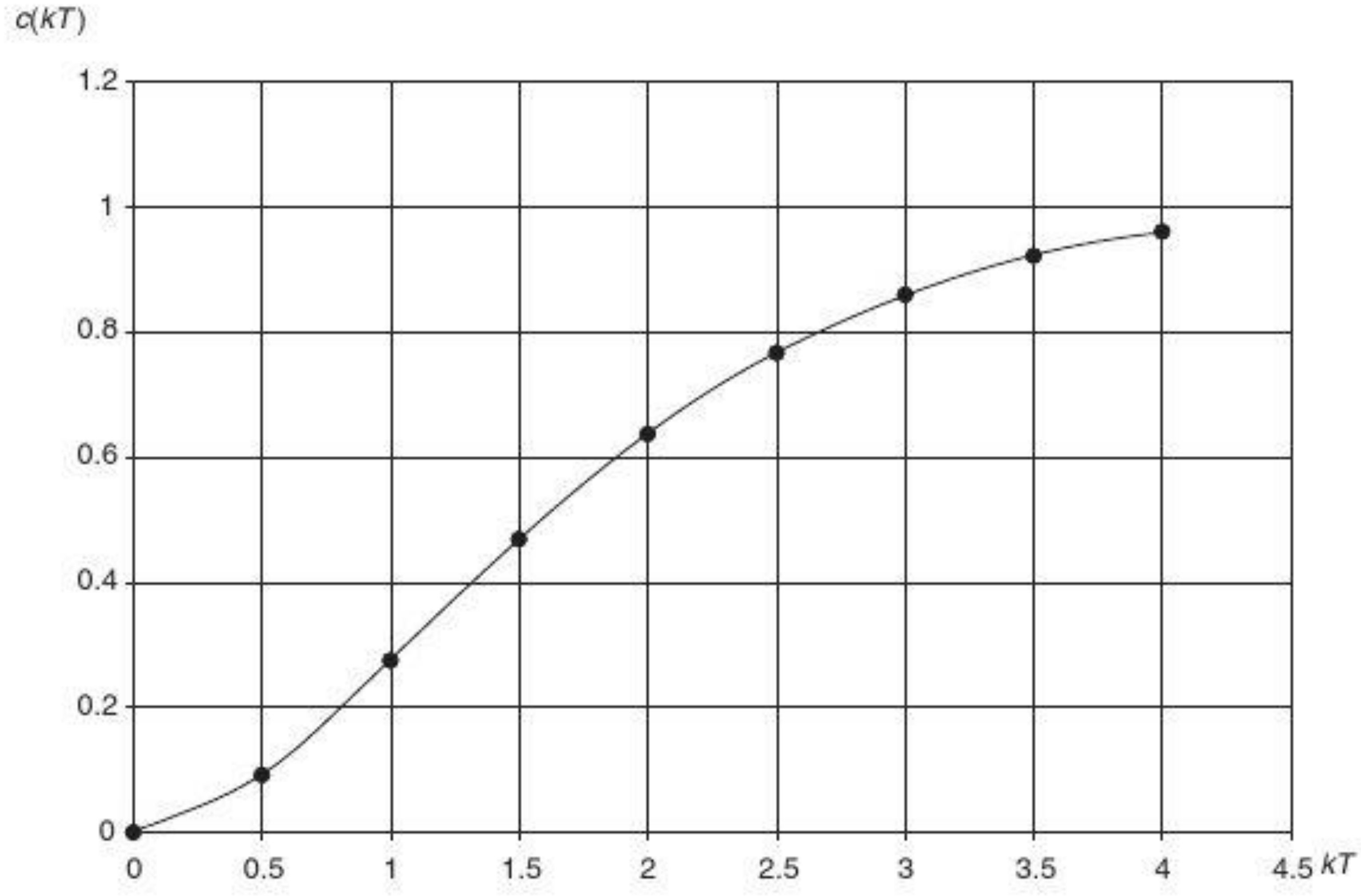

Fig. 7.15 Unit step response for Example 7.4.

$$c(\infty) = \left(\frac{0.092 + 0.066}{1 - 1.276 + 0.434}\right) = 1.0 \tag{7.60}$$

Hence there is no steady-state error.

7.6 Stability in the *z*-plane

7.6.1 Mapping from the *s*-plane into the *z*-plane

Just as transient analysis of continuous systems may be undertaken in the s-plane, stability and transient analysis on discrete systems may be conducted in the z-plane.

It is possible to map from the s to the z-plane using the relationship

$$z = e^{sT} \tag{7.61}$$

now

$$s = \sigma \pm j\omega$$

therefore

$$z = e^{(\sigma \pm j\omega)T} = e^{\sigma T} e^{j\omega T} \quad \text{(using the positive } j\omega \text{ value)} \tag{7.62}$$