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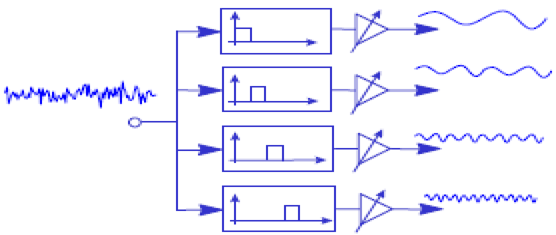
University Of Diyala – College Of Engineering

Digital Signal Processing



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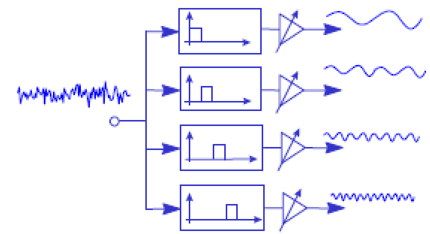

Lecture No. 8: Digital Filters-IIR

Third Class

Department of Computers and Software Engineering

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Lecture Outline

- Infinite Impluse Response of Digital Filters.
- Methods for Calculation of Suitable Filter Coefficients.
- Pole-Zero Placement Method.
- Impulse Invariant Method.
- Matched Z-Transform Method.
- Bilinear Z-transformation Method.
- Realization Structures for IIR Digital Filters .

Infinite Impulse Response Digital filters

Definition: Infinite Impulse Response

- IIR Digital filters are characterized by the following recursive equation

$$y[n]=\sum_{k=0}^{\infty}h[k]x[n-k]$$

$$=\sum_{k=0}^{N}b_k x[n-k]-\sum_{k=1}^{M}a_k y[n-k] \longrightarrow \quad (1)$$

  - h(k) – impulse response of the filter which is theoretically infinite in duration
  - bk & ak – coefficients of the filter
  - x(n) – input to the filter & y(n) – output of the filter

- Transfer function of IIR filter is

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-N}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Mz^{-M}}=\frac{\sum_{k=0}^{M}b_kz^{-k}}{1+\sum_{k=1}^{N}a_kz^{-k}} \longrightarrow \quad (2)$$

Infinite Impulse Response Digital Filters

- The important part is to find suitable values of the filter coefficients (ak & bk) so that the filter behaves in the desired manner.
- Alternative names: pole-zero filters & autoregressive moving average filters.
- In eq:1 the o/p y(n) is a function of the past o/ps, y(n-k) as well as present & past i/p samples, x(n) & x(n-k), i-e the IIR filter is a feedback system of some sort.
- The transfer function of IIR filter, H(z) given in eq:2 can be factored as

$$H(z) = \frac{k(z-z_1)(z-z_2)\ldots(z-z_N)}{(z-p_1)(z-p_2)\ldots(z-p_M)}$$

  - Where z1,z2…are the zeros of H(z) for which H(z) becomes zero, and p1,p2…are poles of H(z), for which H(z) is infinite.

- By plotting these poles and zeros of the transfer function we can analyze the filter in the complex z-plane.
- For the filter to be stable, all its poles must lie inside the unit circle
- There is no restriction on the zeros locations.

Methods for Calculation of Suitable Filter Coefficients

- The task is to calculate the values of the filter coefficients ak & bk.
- There are four methods
  1. Pole-zero placement
  2. Impulse invariant
  3. Matched z transform
  4. Bilinear z-transformation
- The most simple method is pole-zero placement and is used to design very simple filters.
- A more efficient approach is first to design an analog filter & then to convert it into an equivalent digital filter.
- Adv: of this efficient approach is that info: on analog filter designing is already present which can be utilized.
- 2nd, 3rd and 4th methods are based on the efficient approach.

Pole-Zero Placement Method

- When a zero is placed at a given point on the z-plane, the freq: response will be zero at the corresponding point.
- When a pole is placed it produces a peak at the corresponding freq: point (see figure on next slide)
- Poles that are close to the unit circle give rise to large peaks.
- Zeros close to or on the circle produces troughs/minima.
- Thus, by strategically placing poles & zeros on the z-plane, we can obtain simple LP or other freq: selective filters (like HP, BP & BS).

Pole-Zero Placement Method

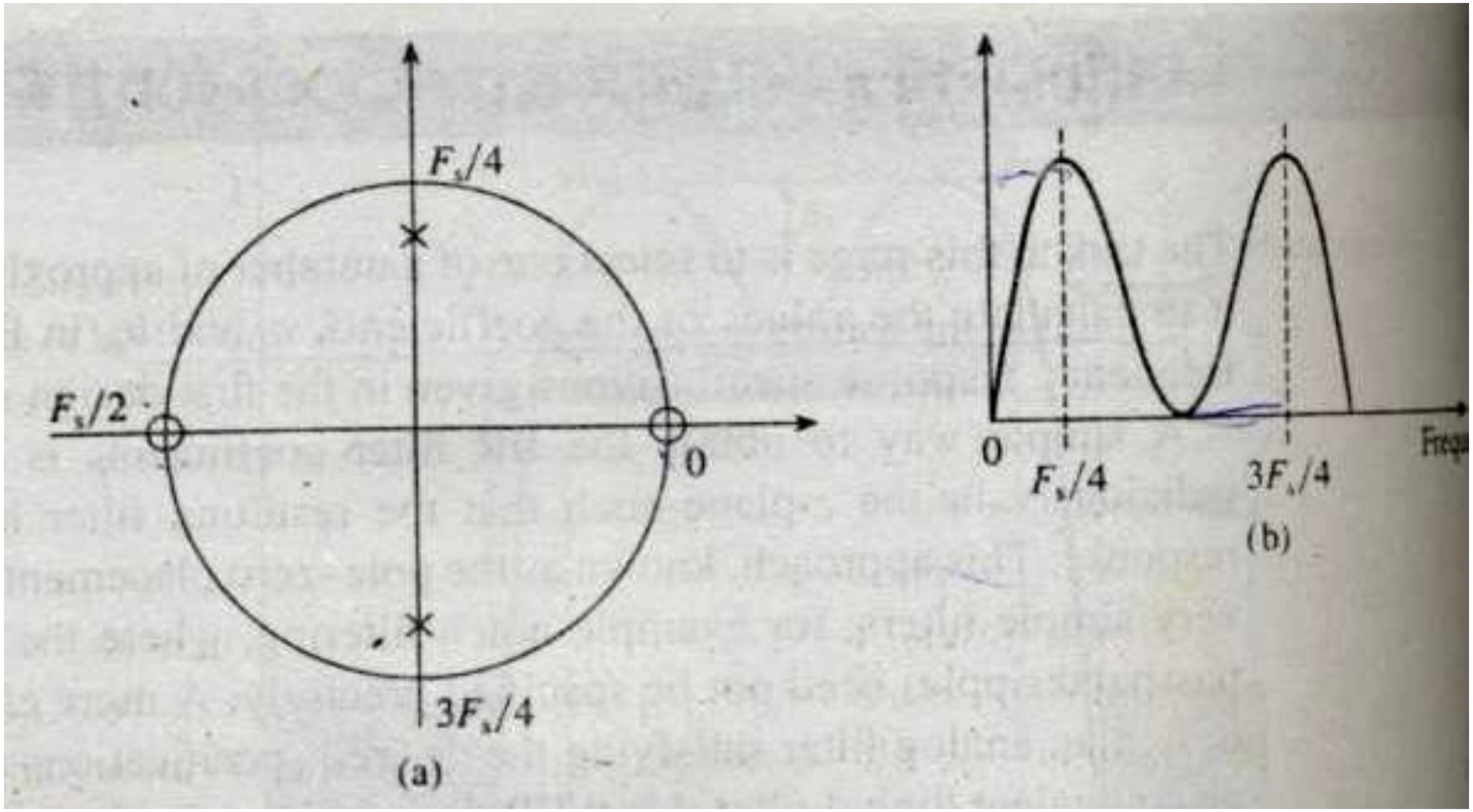
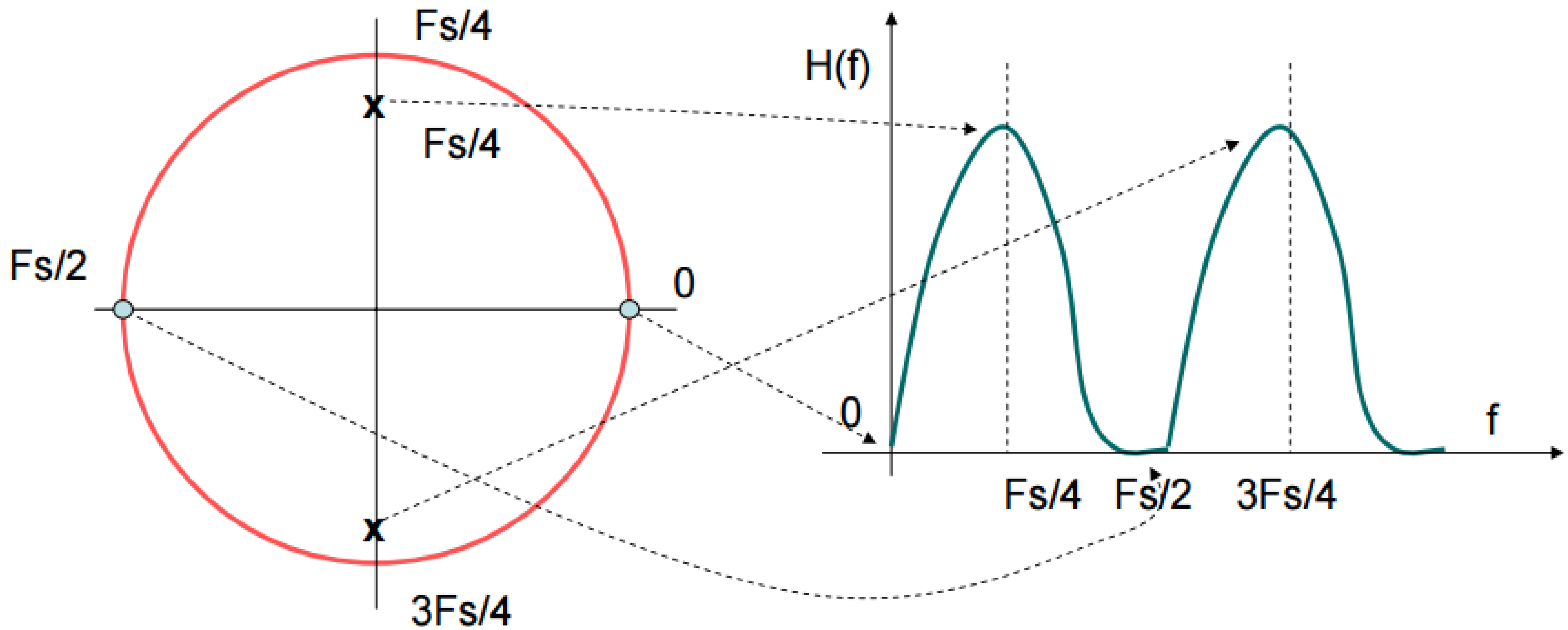


Figure: (a) A pole zero diagram of a simple filter. &
(b) sketch of its frequency response.

Pole-Zero Placement Method

Zero in the z- plane → H(f)=0

Pole in the z- plane → H(f)=max

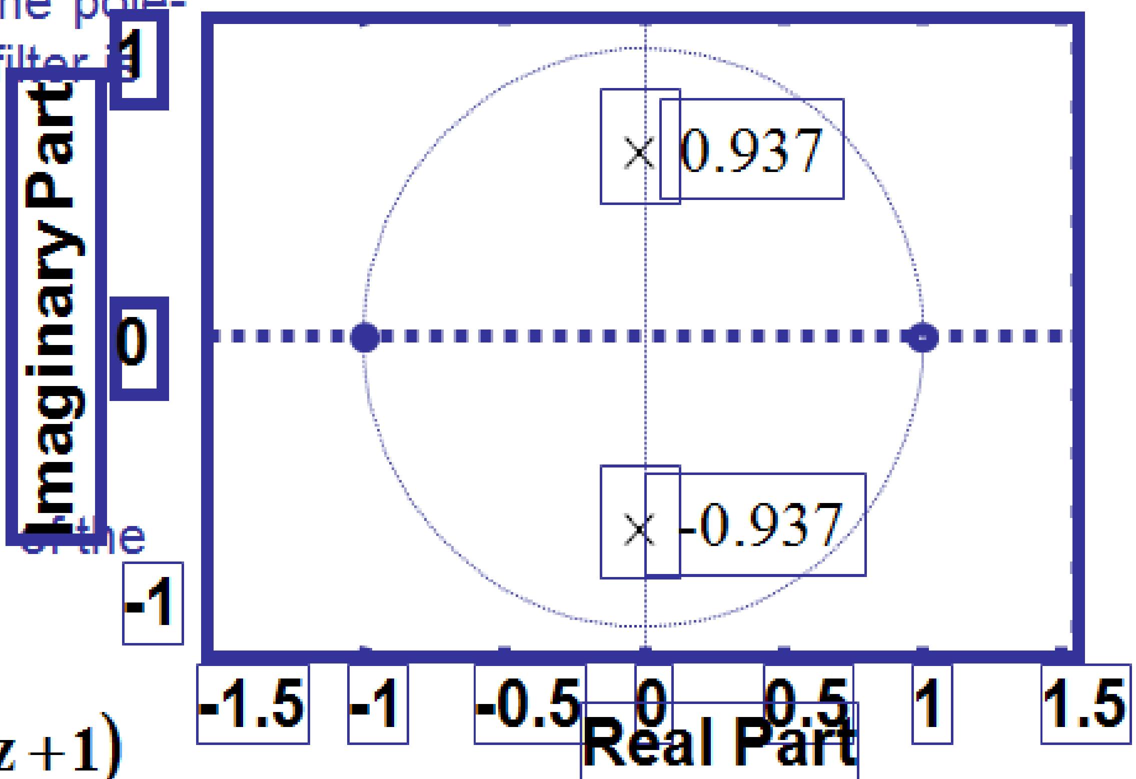


Example1

- Example 1: A bandpass digital filter is required to meet the following specifications:
  (1) complete signal rejection at dc and 250 Hz
  (2) a narrow passband centered at 125 Hz
  (3) a 3 dB bandwidth of 10 Hz
  Assuming a sampling frequency of 500 Hz, obtain the transfer function of the filter, by suitably placing z-plane poles and zeros, and its difference equations
- Solution: First, we must determine where to place the poles and zeros on the z-plane. Since a complete rejection is required at 0 and 250 Hz, we need to place zeros at corresponding points on the z-plane. These are at angles of 0^0 and $360^0 \times 250/500 = 180^0$ on the unit circle. To have the passband centered at 125 Hz requires us to place poles at $\pm 360^0 \times 125/500 = \pm 90^0$. The radius r of the poles is determined by the desired bandwidth. An approximate relation between r and bandwidth, bw, is given by $r \approx 1-(bw/Fs)\pi$

Example1

- For this problem, r = 1 - (10/500)π = 0.937. The pole-zero diagram of the filter is given below.



- The transfer function of the filter is

$$H(z) = \frac{(z-1)(z+1)}{\left(z - re^{j\pi/2}\right)\left(z - re^{-j\pi/2}\right)}$$

Example1

$$= \frac{z^2 - 1}{z^2 + 0.877969} = \frac{1 - z^{-2}}{1 + 0.877969 z^{-2}}$$

The difference equation is

$y[n] = -0.877969y[n-2] + x[n] – x[n-2]$

The coefficients of the filter are therefore given by

b0 = 1, b1 = 0, b2 = -1

a1 = 0, a2 = 0.877969

Impulse Invariant method

Basic Concept:

- In this method, starting with a suitable analog transfer function, H(s), the impulse response, h(t), is obtained using the Laplace transform.
- The h(t) so obtained is suitably sampled to produce h(nT),
- and the desired transfer function, H(z), is then obtained by z-transforming h(nT), where T is the sampling interval.

Steps of Impulse Invariant Method

Following are the main steps of this method.

- Determine a normalized analogue filter, H(s), that satisfies the specifications for the desired digital filter.
- If necessary, expand H(s) using partial fractions to simplify the next step.
- Obtain the z-transform of each partial fraction.
- Obtain H(z) by combining the z-transforms of the partial fractions. If the actual sampling frequency is used then multiply H(z) by T.

Example

- Example: It is required to design a digital filter to approximate the following normalized analog transfer function.

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Using the impulse invariant method obtain the transfer function, H(z), of the digital filter, assuming a 3dB cutoff frequency of 150Hz & a sampling frequency of 1.28kHz.

- Solution: Before applying the impulse invariant method, we need to frequency scale the normalized transfer function. This is achieved by replacing s by s/w_c, where $w_c = 2\times\pi\times150 =$ 942.4778rad/sec, to ensure that the resulting filter has the desired response. Thus

Example

$$H'(s) = H(s)\Big|_{s=\frac{s}{w_c}} = \frac{w_c^2}{s^2 + \sqrt{2}w_c s + w_c^2}$$

$$= \frac{w^2{}_c}{\left(s + \frac{\sqrt{2}w_c}{2}(1+j)\right)\left(s + \frac{\sqrt{2}w_c}{2}(1-j)\right)} = \frac{\frac{w_c}{\sqrt{2}}j}{s + \frac{\sqrt{2}}{2}w_c(1+j)} + \frac{-\frac{w_c}{\sqrt{2}}j}{s + \frac{\sqrt{2}}{2}w_c(1-j)}$$

By taking inverse LT we can get h(t), as

$$h(t) = L^{-1}\{H(s)\}$$

$$= L^{-1}\left\{\frac{c}{s-p}\right\} = ce^{pt}$$

Example

$$h(t)=\frac{w_c}{\sqrt{2}}\,j\left[e^{-\frac{\sqrt{2}}{2}w_c(1+j)t}-e^{-\frac{\sqrt{2}}{2}w_c(1-j)t}\right]$$

Now by sampling h(t) we get h[nT]

$$h[nT]=h(t)\big|_{t=nT}=Ce^{pnT}$$

$$h[nT]=\frac{w_c}{\sqrt{2}}\,j\left[e^{-\frac{\sqrt{2}}{2}w_c(1+j)nT}-e^{-\frac{\sqrt{2}}{2}w_c(1-j)nT}\right]$$

The transfer function of H(z) is obtained by z-transforming h[nT]

$$H(z)=\sum_{n=0}^{\infty}h[nT]z^{-n}=\sum_{n=0}^{\infty}Ce^{pnT}z^{-n}=\frac{C}{1-e^{pT}z^{-1}}$$

$$H(z)=\frac{w_c}{\sqrt{2}}\,j\left[\frac{1}{1-e^{-\frac{\sqrt{2}}{2}w_c(1+j)T}z^{-1}}-\frac{1}{1-e^{-\frac{\sqrt{2}}{2}w_c(1-j)T}z^{-1}}\right]$$

Example

$$= \frac{\sqrt{2} w_c z^{-1} e^{\frac{-\sqrt{2}}{2} w_c T} \sin\left(\frac{\sqrt{2}}{2} w_c T\right)}{1 - 2z^{-1} e^{-\frac{\sqrt{2}}{2} w_c T} \cos\left(\frac{\sqrt{2}}{2} w_c T\right) + z^{-2} e^{-\sqrt{2} w_c T}}$$

$$= \frac{393.9264 \; z^{-1}}{1 - 1.0308 \; z^{-1} + 0.3530 \; z^{-2}}$$

In order to keep the gain down and to avoid overflows, it is common practice to multiply H(z) by T = 1/fs = 1/1280

$$H(z) = \frac{0.3076 z^{-1}}{1 - 1.0308 z^{-1} + 0.3530 z^{-2}}$$

Thus we have, $b_0 = 0$; $b_1 = 0.3076$; $a_1 = -1.0308$; $a_2 = 0.3530$

Matched Z-Transform Method

Basic Concept:

- The MZT method provides a simple way to convert an analog filter into an equivalent digital filter.
- Each of the poles and zeros of the analog filter is mapped directly from s-plane to the z-plane using the following equation. $(s-a) \rightarrow (1-z^{-1}e^{at})$
  - Where T is the sampling period.
- For a higher order analog filters, the transfer function may be written in the form:

$$H(z) = \frac{k(z-z_1)(z-z_2)\dots(z-z_N)}{(z-p_1)(z-p_2)\dots(z-p_M)}$$

  - Where zk & pk are zeros & poles of H(s)

Matched Z-Transform method

The MZT may then be applied to each factor separately:

$$(s - z_k) \rightarrow (1 - z^{-1}e^{z_k T}) \;\&\; (s - p_k) \rightarrow (1 - z^{-1}e^{p_k T})$$

For second order filter, the transfer function reduces to

$$\frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)} \rightarrow \frac{1 - (e^{z_1 T} + e^{z_2 T})z^{-1} + e^{(z_1 + z_2)T} z^{-2}}{1 - (e^{p_1 T} + e^{p_2 T})z^{-1} + e^{(p_1 + p_2)T} z^{-2}}$$

If the poles & zeros occur in complex conju: pairs then the eq: simplifies to:

$$\frac{1 - 2e^{z_r T}\cos(z_i T)z^{-1} + e^{z_r T} z^{-2}}{1 - 2e^{p_r T}\cos(p_i T)z^{-1} + e^{p_r T} z^{-2}}$$

Analog filter in polynomial format can be written as:

$$H(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)} = \frac{A_0 + A_1 s + A_2 s^2}{B_0 + B_1 s + B_2 s^2}$$

The poles & zeros of H(s) are then given by

$$p_{1,2} = \frac{-B_1}{2B_2} \pm \left[\left(\frac{B_1}{2B_2}\right)^2 - \frac{B_0}{B_2}\right]^{\frac{1}{2}} \;\&\; z_{1,2} = \frac{-A_1}{2A_2} \pm \left[\left(\frac{A_1}{2A_2}\right)^2 - \frac{A_0}{A_2}\right]^{\frac{1}{2}}$$

Example

- Example: The normalized transfer function of an analog filter is given by

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Using the MZT method obtain the transfer function, H(z), of the digital filter, assuming a 3dB cutoff frequency of 150Hz & a sampling frequency of 1.28kHz.

- Solution: The cutoff freq: may be expressed as $w_c = 2\times\pi\times150 = 942.4778$ rad/sec. The transfer function analog filter is obtained by replacing s by s/w_c:

$$H'(s) = H(s)\Big|_{s=\frac{s}{w_c}} = \frac{w_c^2}{s^2 + \sqrt{2}w_c s + w_c^2}$$

Example

The poles of the filter are located at

$$p_{1,2} = \frac{-\sqrt{2}\omega_c}{2} \pm \left[\left(\frac{\sqrt{2}\omega_c}{2} \right)^2 - {\omega_c}^2 \right]^{\frac{1}{2}}$$

$$p_{1,2} = \frac{-\sqrt{2}\omega_c}{2} \pm \omega_c \left[\left(\frac{\sqrt{2}}{2} \right)^2 - 1 \right]^{\frac{1}{2}}$$

$$p_{1,2} = \frac{-\sqrt{2}\omega_c}{2} (1 \mp j)$$

For the problem, the real & imaginary parts of the poles are

$$p_r = \frac{-\sqrt{2}\omega_c}{2} = -666.4324, p_i = \frac{\sqrt{2}\omega_c}{2} j = 666.4324j$$

Example

Thus, p_rT = -0.5206503, p_iT = 0.5206503, $\cos(p_iT)$ = 0.867496 & e^{p_rT} = 0.594134.The resulting transfer function becomes

$$H(z) = \frac{\omega_c^2}{1 - 2e^{p_rT}\cos(p_iT)z^{-1} + e^{p_rT}z^{-2}}$$

$$H(z) = \frac{8.8876 \times 10^5}{1 - 1.030818z^{-1} + 0.594134z^{-2}}$$

Bilinear z-transformation method

Basic concept:

- This is the most widely used transformation which is suitable for the design of low-pass, high-pass, band-pass and band-stop filters.
- In the Bilinear Transformation (BLT) method, the basic operation required to convert an analogue filter is to replace s as follows:

$$s = k\frac{z-1}{z+1}, \quad \mathrm{k} = 1 \text{ or } 2/\mathrm{T}$$

- The direct replacement of s in the above eq: may lead to a digital filter with an undesirable response.
- The solution is that, we prewarp the critical frequencies before applying the BZT.
- The prewarp bandedge freq: is given as:

$$\omega_p' = k\tan\left(\frac{\omega_p T}{2}\right)$$

Prewarp cutoff freq: (ω_p')

Specified cutoff freq: (ω_p)

Steps of BLT method

For standard, frequency selective IIR filters, the steps for using the BLT method are:

1. Use the digital filter specifications to find a suitable normalized, prototype, analogue lowpass filter, H(s).
2. Determine and prewarp the bandedge or critical frequencies of the desired filter.
3. Denormalize the analog prototype filter by replacing s in the transfer function H(s), using one of the following transformations, depending on the type of filter required:

Steps of BLT method

$$s = \frac{s}{w'_p} \qquad \text{lowpass to lowpass} \qquad (7)$$

$$s = \frac{w'_p}{s} \qquad \text{lowpass to highpass} \qquad (8)$$

$$s = \frac{s^2 + w_0^2}{Ws} \qquad \text{lowpass to bandpass} \qquad (9)$$

$$s = \frac{Ws}{s^2 + w_0^2} \qquad \text{lowpass to bandstop} \qquad (10)$$

where $Ws = w'_{p2} - w'_{p1}$ and $\mathbf{w_0^2 = w'_{p2} w'_{p1}}$

4. Apply the BLT to obtain the desired digital filter transfer func:

$$s = \frac{z-1}{z+1},$$

Example

- **Example**: Design a digital low-pass filter to approximate the following transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Using the BLT method obtain the transfer function, H(z), of the digital filter, assuming a 3 dB cutoff frequency of 150 Hz and a sampling frequency of 1.28kHz.

- **Solution:** $w_p = 2\pi \times 150$ rad/sec, $T = 1/F_s = 1/1280$, giving a prewarped critical frequency of $w'_p = \tan(w_p T/2) = 0.3857$.

The frequency scaled analog filter is given by

$$H'(s) = H(s)\big|_{s=s/\omega'_p} = \frac{1}{\left(s/w'_p\right)^2 + \sqrt{2}(s/w'_p) + 1} = \frac{(w'_p)^2}{s^2 + \sqrt{2}sw'_p + (w'_p)^2}$$

Example

Applying the BLT gives

$$H(z) = H'(s)\Big|_{s=\frac{z-1}{z+1}} = \frac{0.0878\,z^2 + 0.1756\,z + 0.0878}{z^2 - 1.0048\,z + 0.3561}$$

$$= \frac{0.0878\left(1 - 2z^{-1} + z^{-2}\right)}{1 - 1.0048\,z^{-1} + 0.3561\,z^{-2}}$$

Realization Structures for IIR Digital Filters

(a) Direct Form Realizations:

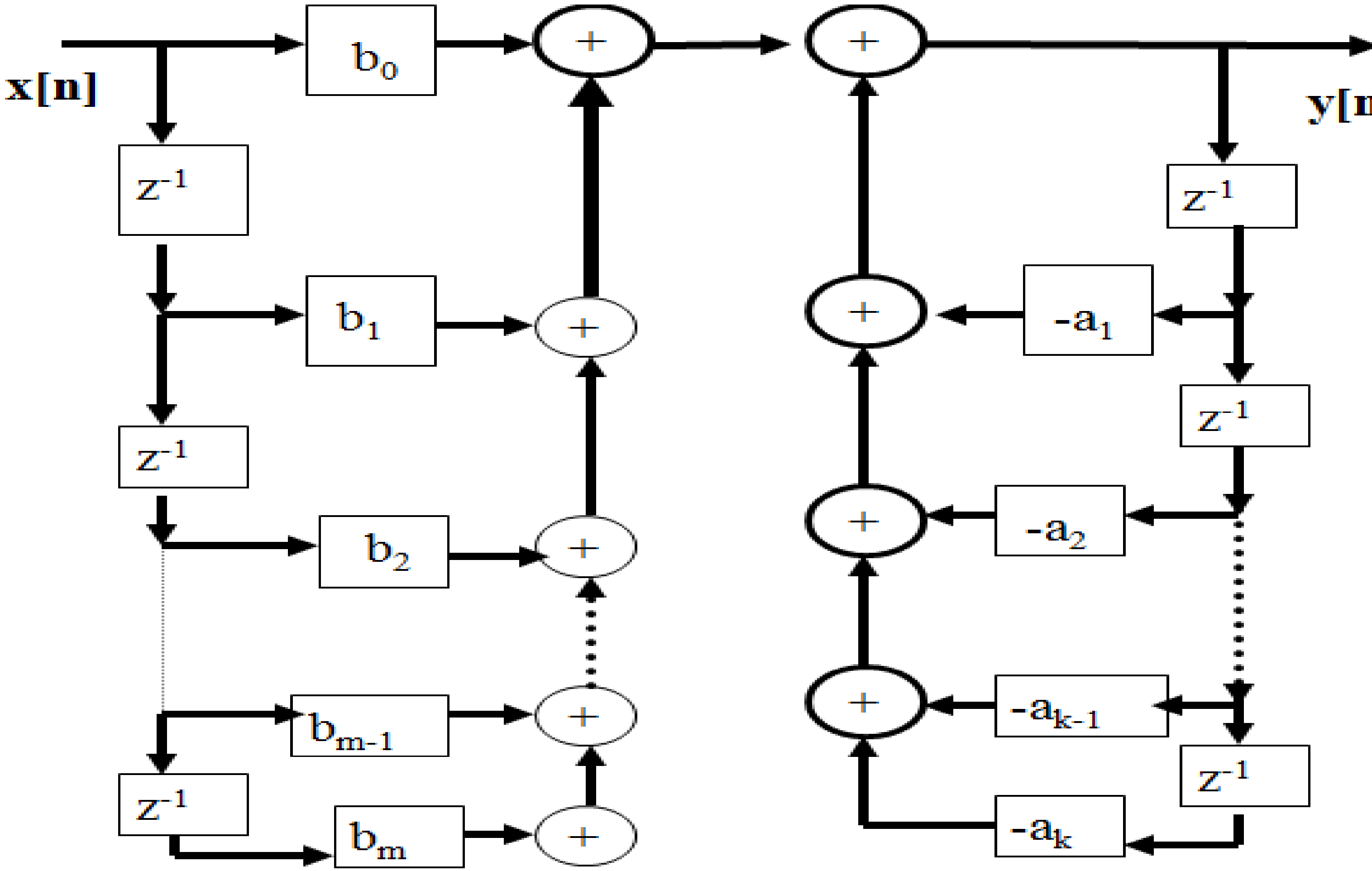
(i) Direct form 1

We generate what is called a *Direct Form I Realization* by directly implementing the system difference equation

$$\mathbf{y[n]} = \sum_{\mathbf{i=1}}^{\mathbf{k}} (-\mathbf{a_i})\mathbf{y[n-i]} + \sum_{\mathbf{i=0}}^{\mathbf{m}} (\mathbf{b_i})\mathbf{x[n-i]} \qquad (1)$$

The result is shown in Fig.

(i) Direct Form I

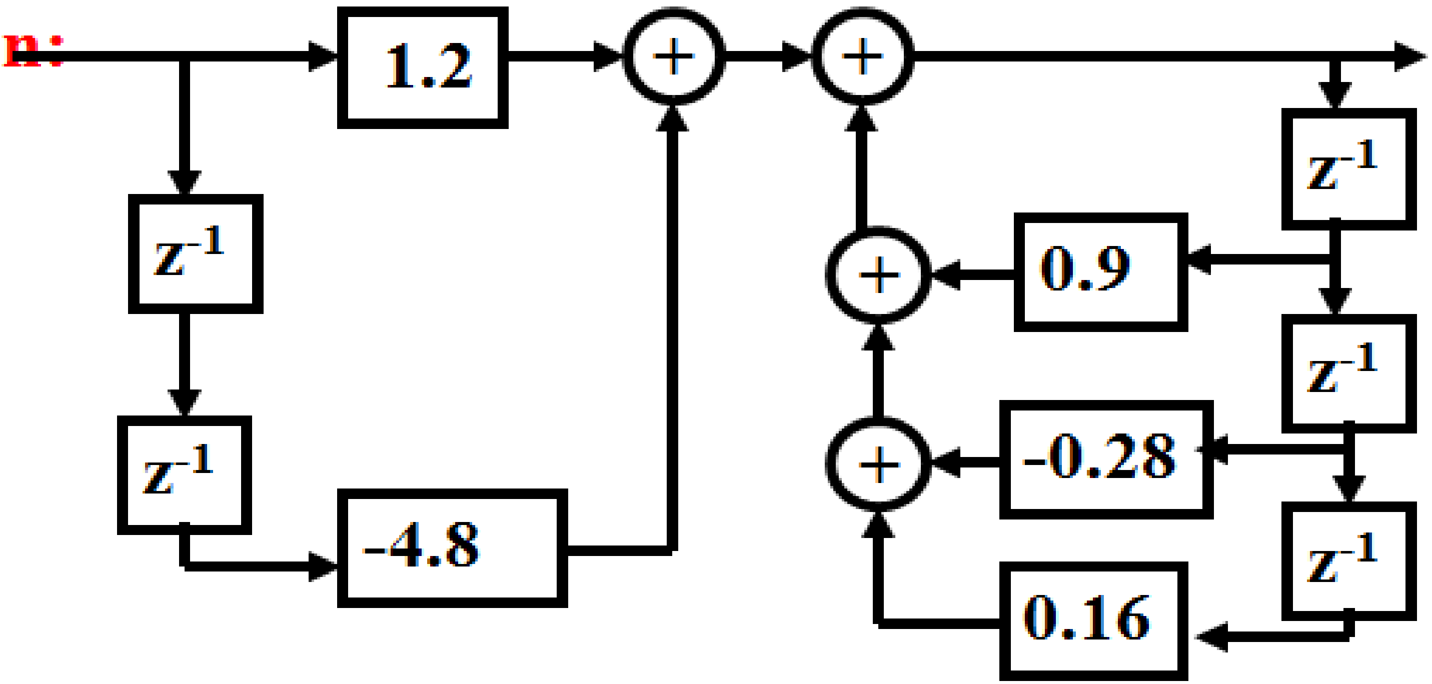


(i) Direct Form I

Example: Draw the block diagram representation for Direct Form 1 of the system having the following transfer function:

$$H(z) = \frac{1.2 - 4.8z^{-2}}{1 - 0.9z^{-1} + 0.28z^{-2} - 0.16z^{-3}}$$

Solution:



(ii) Direct Form II

The z-transform of the system transfer function that correspond to difference equation (1) is

$$H(z) = \frac{\sum_{i=0}^{m} b_i z^{-i}}{1 + \sum_{i=1}^{k} a_i z^{-i}} \qquad (2)$$

To generate the Direct Form II realization, we re-write (2) as

$$Y(z) = H(z)X(z) = \frac{\sum_{i=0}^{m} b_i z^{-i}}{1 + \sum_{i=1}^{k} a_k z^{-i}} X(z) = \left[\sum_{i=0}^{m} b_i z^{-i}\right] W(z) \qquad (3)$$

where

$$W(z) = \frac{X(z)}{1 + \sum_{i=1}^{k} a_i z^{-i}} \qquad (4)$$

We then re-write (4) in the form

(ii) Direct Form II

$$W(z) = X(z) - \sum_{i=1}^{k} (a_i) z^{-i} W(z)$$

Computation of the inverse z-transforms of W(z) yields

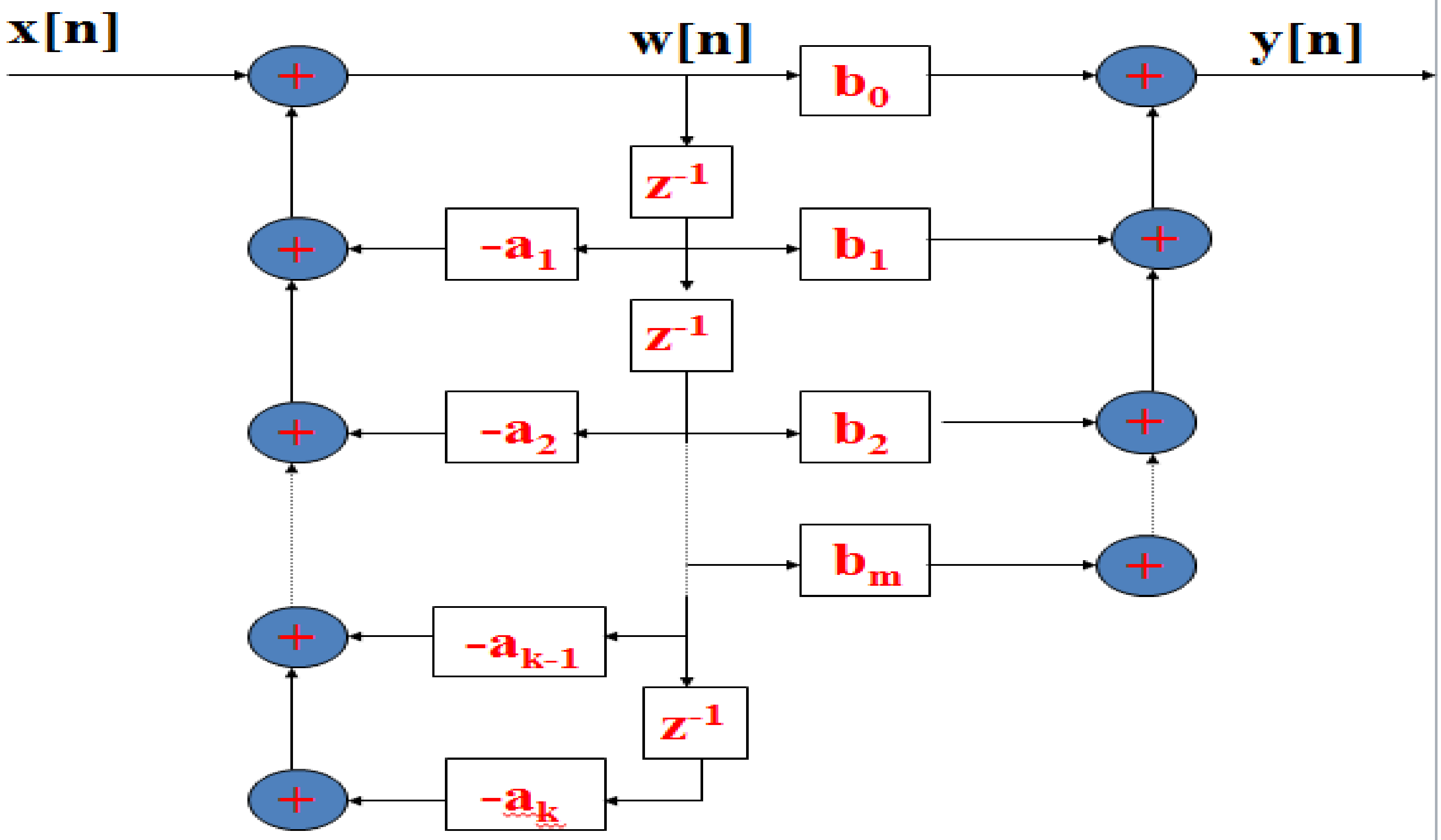
$$w[n] = x[n] + \sum_{i=1}^{k} (-a_i) w[(n-i)] \qquad (5)$$

and

$$y[n] = \sum_{i=0}^{m} b_i w[n-i] \qquad (6)$$

Equations (5) and (6) define the Direct Form II Realization represented by the block diagram Shown in the next slide.

(ii) Direct Form II

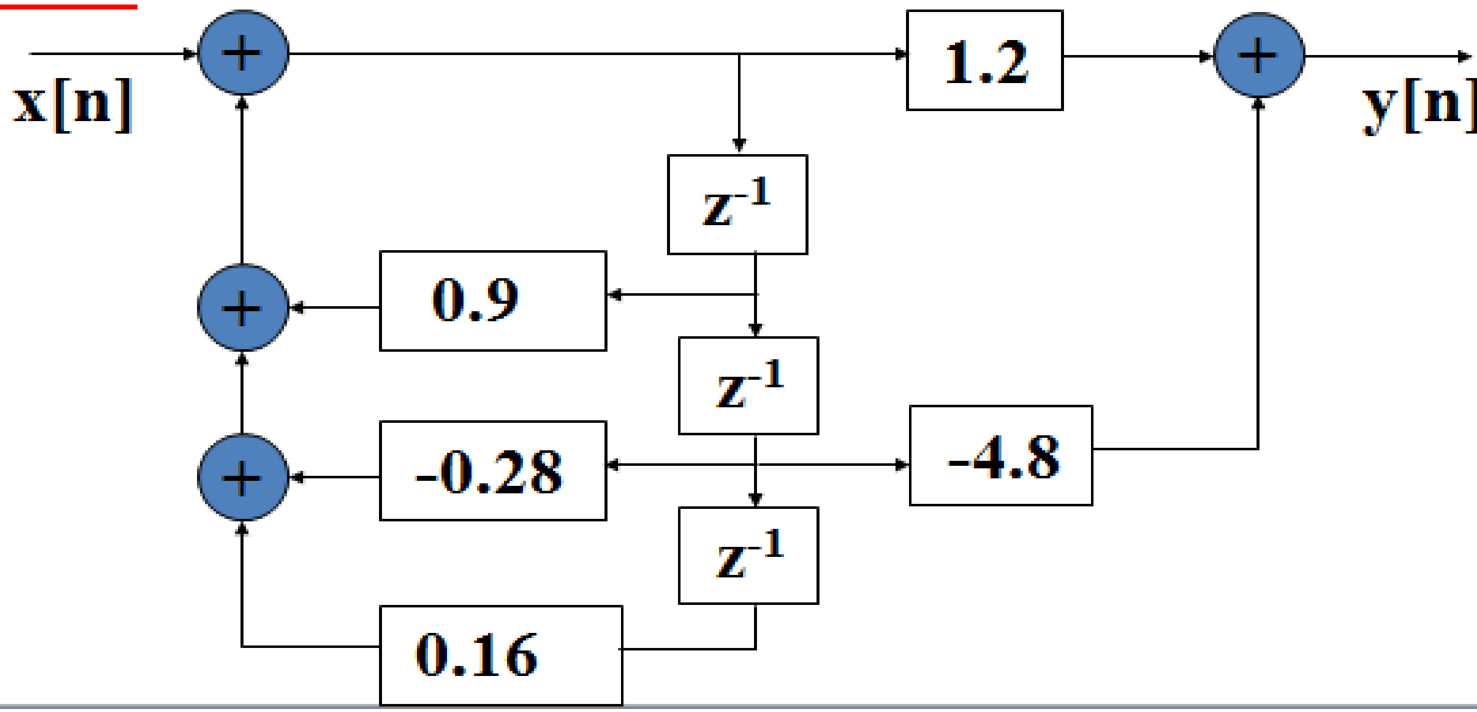


Example

Example: Draw the block diagram representation for Direct Form II realization of the system having transfer function:

$$\frac{1.2 - 4.8z^{-2}}{1 - 0.9z^{-1} + 0.28z^{-2} - 0.16z^{-3}}$$

Solution:



Cascade Realisation

We illustrate this realization with the help of an example:

Example: Find a cascade realization of the system characterized by the transfer function

$$H(z)=\frac{4z^3+16z^2+4z-24}{2z^4+1.6z^3+0.5z^2+0.1z}$$

Solution: We factor the numerator and denominator to obtain

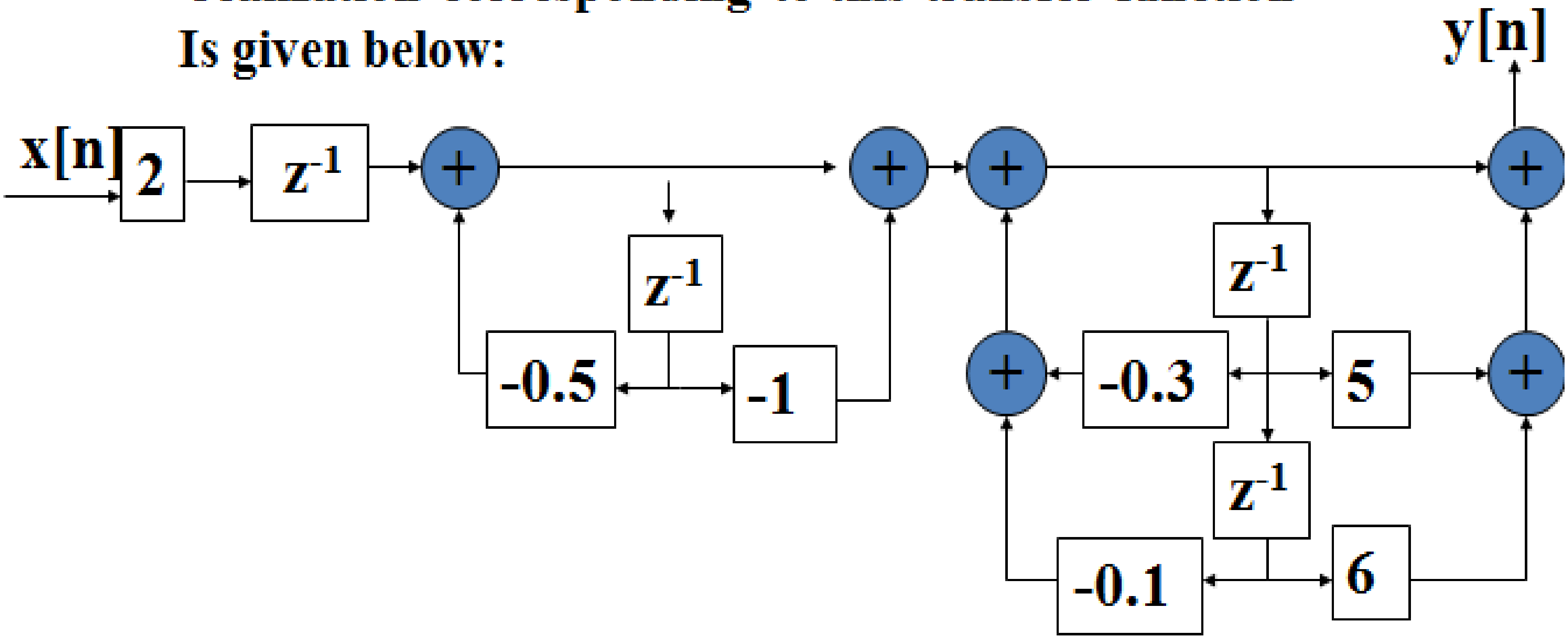
$$H(z)=\frac{2(z-1)(z+2)(z+3)}{z(z+0.5)(z^2+0.3z+0.1)}$$

or

$$H(z)=2(z^{-1})\left(\frac{1-z^{-1}}{1+0.5z^{-1}}\right)\left(\frac{1+5z^{-1}+6z^{-2}}{1+0.3z^{-1}+0.1z^{-2}}\right)$$

Cascade Realisation

The block diagram representation of the cascade realization corresponding to this transfer function Is given below:



Parallel Realisation

Example: Find the parallel realization of the system of

$$H(z) = \frac{4z^3 + 16z^2 + 4z - 24}{2z^4 + 1.6z^3 + 0.5z^2 + 0.1z}$$

Solution: Multiply the numerator and denominator by 0.5 and factor the denominator. The result is

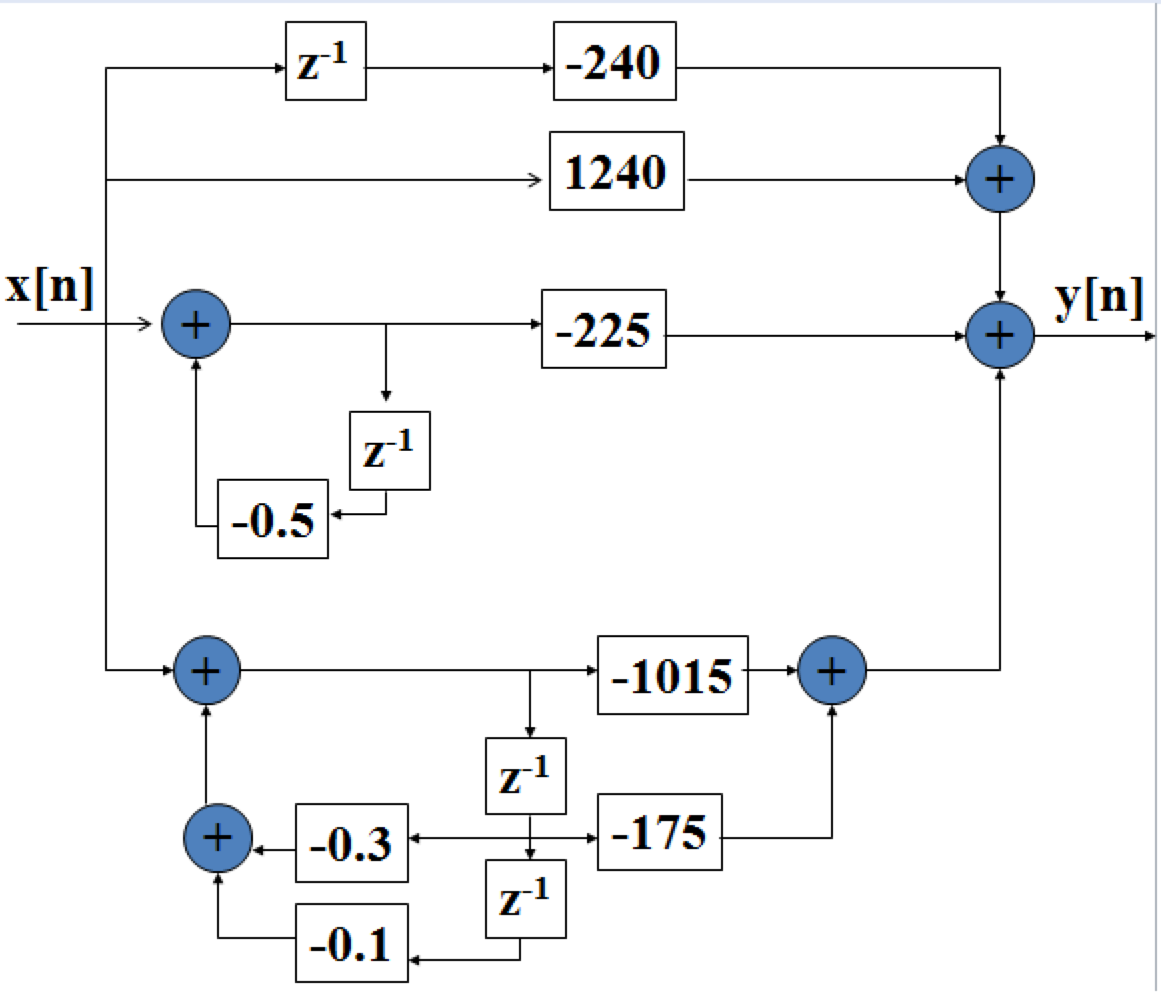
$$H(z) = \frac{2z^3 + 8z^2 + 2z - 12}{z(z + 0.5)(z^2 + 0.3z + 0.1)}$$

The partial fraction expansion is

$$H(z) = -240z^{-1} + 1240 + \frac{-225}{1 + 0.5z^{-1}} + \frac{-1015 - 175z^{-1}}{1 + 0.3z^{-1} + 0.1z^{-2}}$$

Now we can realize the system as shown on the next slide.

Parallel Realisation



Finite word length effects in IIR filters

- ADC quantization noise:
- Coefficients quantization errors
- Overflow errors
- The following components are needed for the implementation of IIR filters.
  - Memory (for eg: ROM) for filter coefficients
  - Memory (RAM)
  - Hardware or software
  - Adder or arithmetic logic unit.
  - In modern real-time DSP processor such as the TMS320C50
  - 8-bit or 16-bit MPU such as the motorola 6800 or 68000

Tutorial

Question1: Design LP & HP filters using pole –zero placement method.

Question2: The normalized transfer function of a simple, analog lowpass, filter given by

$$H(s) = \frac{1}{s+1}$$

Determine, using the BLT method, the transfer function of an equivalent discrete time high-pass filter. Assume a sampling frequency of 150 Hz and a cutoff frequency of 30 Hz.

Question3: A discrete time bandpass filter with Butterworth characteristics meeting the specifications given below is required. Obtain the coefficients of the filter using the BLT method.

passband	200 – 300 Hz
sampling frequency	2 kHz
filter order	2

End of Chapter