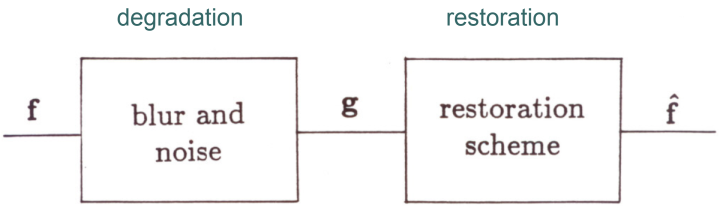
EE421/521
Image Processing

Lecture 11b
IMAGE RESTORATION

Introduction to Image Restoration

Image Restoration



degradation

restoration

f

blur and noise

g

restoration scheme

$\hat{f}$

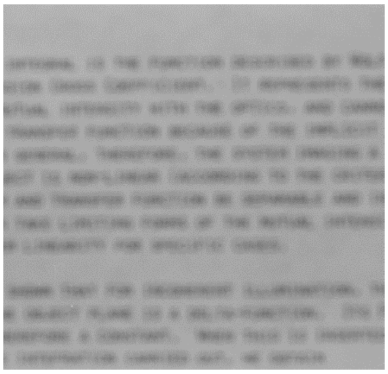
f: original object

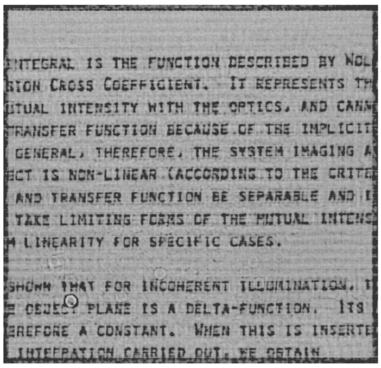
g: distorted noisy image

$\hat{f}$: restored image

Restoration Example

Restoration Example





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INTEGRATION CARRIED OUT, WE OBTAIN

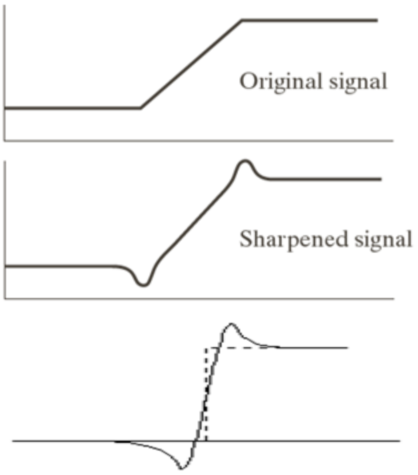
Applications

- Law enforcement
- Medical Imaging
- Space explorations
- Commercial and consumer imaging

Restoration vs. Enhancement

- **RESTORATION**: Undo or invert a mathematical model of the degradation to obtain the ``ideal" image; e.g., Wiener filtering, constrained least-squares, iterative least-squares, POCS, etc.
- **ENHANCEMENT**: Produce a more pleasing image without using a particular model of the degradation; e.g., sharpening by high-frequency emphasis, contrast adjustment by histogram equalization.
- **WHY IMAGE RESTORATION ?**
  Most serious image degradations are caused by distortion of the Fourier phase of the image. Need to model the blurring process accurately in order to be able to correct for the phase distortions.

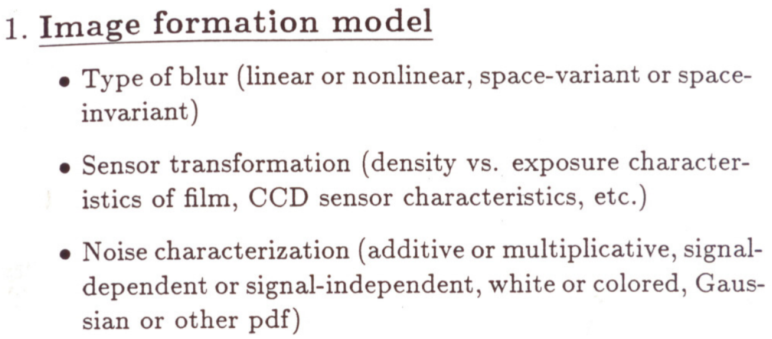
Resolution vs Sharpness



Enhanced signal:
increased **sharpness**

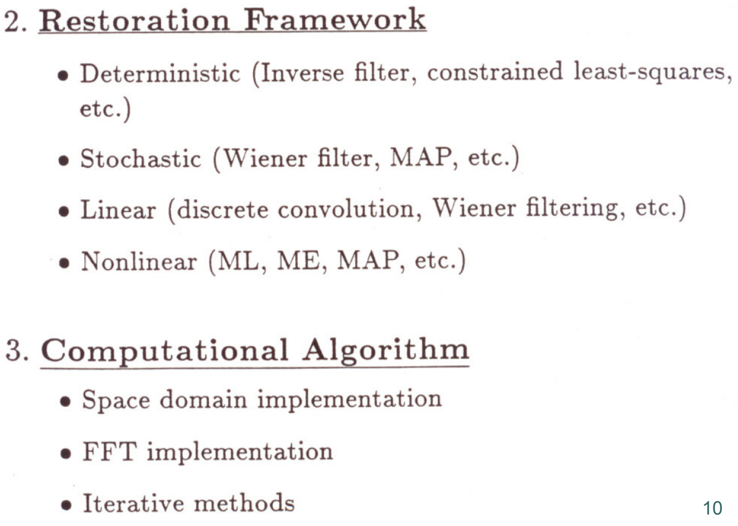
Restored signal:
increased **resolution**

Image Restoration Requirements



1. **Image formation model**
   - Type of blur (linear or nonlinear, space-variant or space-invariant)
   - Sensor transformation (density vs. exposure characteristics of film, CCD sensor characteristics, etc.)
   - Noise characterization (additive or multiplicative, signal-dependent or signal-independent, white or colored, Gaussian or other pdf)

Image Restoration Requirements



2. **Restoration Framework**
   - Deterministic (Inverse filter, constrained least-squares, etc.)
   - Stochastic (Wiener filter, MAP, etc.)
   - Linear (discrete convolution, Wiener filtering, etc.)
   - Nonlinear (ML, ME, MAP, etc.)

3. **Computational Algorithm**
   - Space domain implementation
   - FFT implementation
   - Iterative methods

Image Degradation Model

(Space and Fourier Domain)

Space domain:

$$g(n_1,n_2) = h(n_1,n_2) ** s(n_1,n_2) + v(n_1,n_2)$$

Fourier domain:

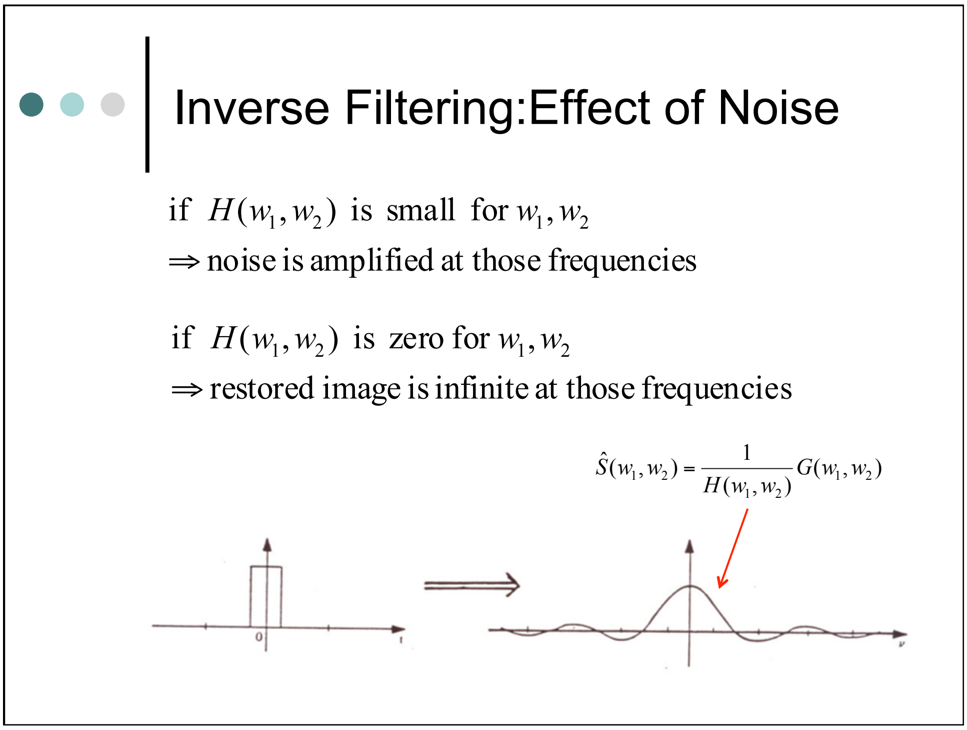
$$G(w_1,w_2) = H(w_1,w_2)S(w_1,w_2) + V(w_1,w_2)$$

Inverse Filtering

$$G(w_1,w_2) = H(w_1,w_2)S(w_1,w_2) + V(w_1,w_2)$$

- Use mathematical inverse of the blur function to restore the original image

$$\hat{S}(w_1,w_2) = \frac{1}{H(w_1,w_2)} G(w_1,w_2)$$
$$= S(w_1,w_2) + \frac{V(w_1,w_2)}{H(w_1,w_2)}$$



Inverse Filtering:Effect of Noise

if $H(w_1,w_2)$ is small for w_1,w_2

$\Rightarrow$ noise is amplified at those frequencies

if $H(w_1,w_2)$ is zero for w_1,w_2

$\Rightarrow$ restored image is infinite at those frequencies

$$\hat{S}(w_1,w_2)=\frac{1}{H(w_1,w_2)}G(w_1,w_2)$$

Pseudo Inverse Filtering

- Use pseudo-inverse to overcome the infinities at the zeroes of the blur

$$\hat{S}(w_1,w_2)=\begin{cases}\dfrac{G(w_1,w_2)}{H(w_1,w_2)} & H(w_1,w_2)\neq 0\\ 0 & H(w_1,w_2)=0\end{cases}$$

Still have noise amplification where $H(w_1,w_2)$ is small.

Optimum LTI Filter for Restoration: Wiener Filter

- Recall the Wiener filter for the blur-free case:

$$\hat{S}(w_1,w_2)=\frac{P_{ss}(w_1,w_2)}{P_{ss}(w_1,w_2)+P_{vv}(w_1,w_2)}G(w_1,w_2)$$

- When there is blur we have

$$s(n_1,n_2)\to h(n_1,n_2)**s(n_1,n_2)$$

- Thus, substitute in the blur-free solution

$$\hat{S}(w_1,w_2)\to H(w_1,w_2)\hat{S}(w_1,w_2)$$
$$P_{ss}(w_1,w_2)\to |H(w_1,w_2)|^2 P_{ss}(w_1,w_2)$$

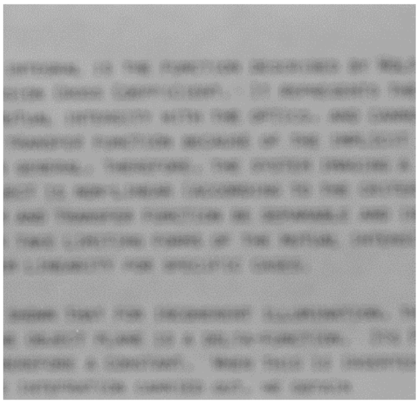
Optimum LTI Filter for Restoration: Wiener Filter

$$H(w_1,w_2)\hat{S}(w_1,w_2)=\frac{|H(w_1,w_2)|^2 P_{ss}(w_1,w_2)}{|H(w_1,w_2)|^2 P_{ss}(w_1,w_2)+P_{vv}(w_1,w_2)}G(w_1,w_2)$$

$$\Downarrow$$

$$\boxed{\hat{S}(w_1,w_2)=\frac{H^*(w_1,w_2)}{|H(w_1,w_2)|^2+\dfrac{P_{vv}(w_1,w_2)}{P_{ss}(w_1,w_2)}}G(w_1,w_2)}$$

Example:
Out of Focus Restoration



Out-of-focus text image

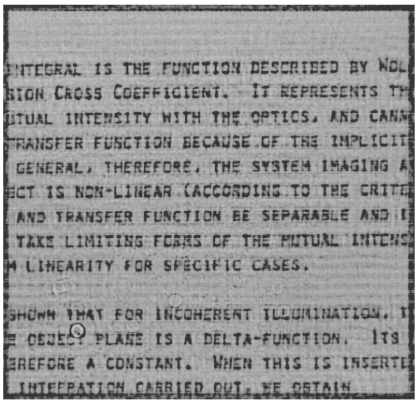


Image restored by Wiener Filtering

Inverse vs. Wiener Filter

inverse filter

Wiener filter

Characteristics of the Wiener Filter

$$\hat{S}(w_1,w_2) = \frac{H^*(w_1,w_2)}{\left|H(w_1,w_2)\right|^2 + \dfrac{P_v(w_1,w_2)}{P_s(w_1,w_2)}} G(w_1,w_2)$$

Noise power

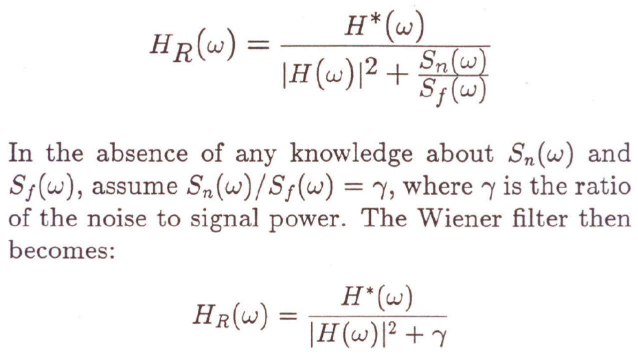
Signal power

- There is no ill-conditinoned behaviour associated with the Wiener filter
- If the noise power is zero at some frequency → we have the inverse filter
- If the signal power is zero at some frequency →the filter becomes zero → we can't recover information at those frequencies where the noise was completely dominant

Shortcomings of the Wiener Filter

- The MMSE (minimum mean squared error) estimate is based on linear assumptions. But there are nonlinearities in the image recording and the human visual system.
- The MMSE is not the criterion that the human visual system naturally employs. MMSE restorations in low SNR cases appear too smooth; the human eye is often willing to accept more visual noise in exchange for additional image structure in the process.

Example



$$H_R(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \frac{S_n(\omega)}{S_f(\omega)}}$$

In the absence of any knowledge about $S_n(\omega)$ and $S_f(\omega)$, assume $S_n(\omega)/S_f(\omega) = \gamma$, where γ is the ratio of the noise to signal power. The Wiener filter then becomes:

$$H_R(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \gamma}$$

- 7x7 blur, 8-bit quantization $\gamma = 10^{-2}$ works best
- 7x7 blur, 12-bit quantization $\gamma = 10^{-4}$ works best

Image Restoration Artifacts

Restoration Artifacts

- Filtered-noise artifacts
- Filter-deviation (regularization) artifacts
- Boundary-truncation artifacts
- PSF-error artifacts

Filtered Noise and Regularization Artifacts

- Let $\Phi(w_1,w_2)$ be some LSI restoration filter.

$$\hat{S}(w_1,w_2) = \Phi(w_1,w_2)G(w_1,w_2) \leftarrow \boxed{G(w_1,w_2) = H(w_1,w_2)S(w_1,w_2) + V(w_1,w_2)}$$

$$= \Phi(w_1,w_2)H(w_1,w_2)S(w_1,w_2) + \Phi(w_1,w_2)V(w_1,w_2)$$

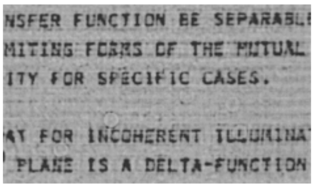
add and subtract $S(w_1,w_2)$, and rearrange terms

$$\hat{S}(w_1,w_2) = S(w_1,w_2) + [\Phi(w_1,w_2)H(w_1,w_2) - 1]S(w_1,w_2) + \Phi(w_1,w_2)V(w_1,w_2)$$

- In the image domain

$$s(\mathbf{x}) = s(\mathbf{x}) + \delta_r(\mathbf{x}) \underbrace{**}_{\text{regularization artifacts}} s(\mathbf{x}) + \phi(\mathbf{x}) \underbrace{**}_{\text{filtered noise artifacts}} v(\mathbf{x})$$

Restoration Artifacts



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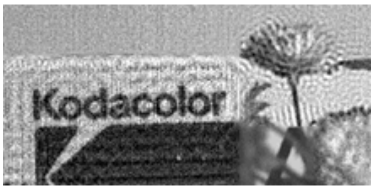
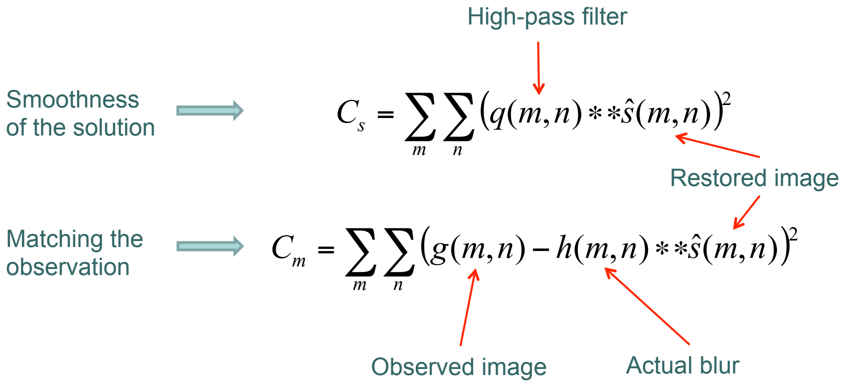


General Linear Regularization: Constrained Least Squares Filter



High-pass filter

Smoothness of the solution

$$C_s = \sum_m \sum_n \left(q(m,n) ** \hat{s}(m,n)\right)^2$$

Restored image

Matching the observation

$$C_m = \sum_m \sum_n \left(g(m,n) - h(m,n) ** \hat{s}(m,n)\right)^2$$

Observed image

Actual blur

Constrained Least Squares Filter

Relative weight

$$\text{minimize} \quad C_m + \overset{\downarrow}{\gamma}\, C_s \quad \text{with respect to} \quad \hat{s}(m,n)$$

Due to Parseval's relation ⟹

$$C_s = \sum_k \sum_l \left(Q(k,l)\hat{S}(k,l)\right)^2$$

$$C_m = \sum_k \sum_l \left(G(k,l) - H(k,l)\hat{S}(k,l)\right)^2$$

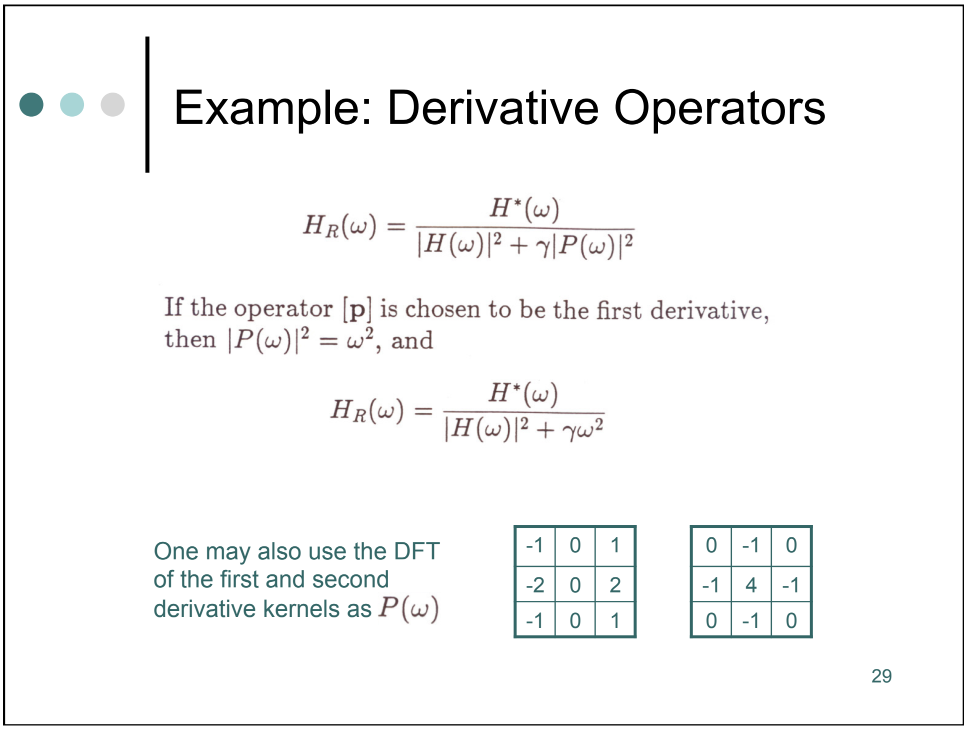
Constrained Least Squares Filter

Minimize

$$C_m + \gamma\, C_s = \sum_k \sum_l \left(\left(G(k,l) - H(k,l)\hat{S}(k,l)\right)^2 + \gamma\left(Q(k,l)\hat{S}(k,l)\right)^2\right)$$

with respect to $\hat{S}(\mathrm{k,l})$

$$\hat{S}(k,l) = \boxed{\frac{H(k,l)}{\left|H(k,l)\right|^2 + \gamma\left|Q(k,l)\right|^2}}\, G(k,l)$$



Example: Derivative Operators

$$H_R(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \gamma|P(\omega)|^2}$$

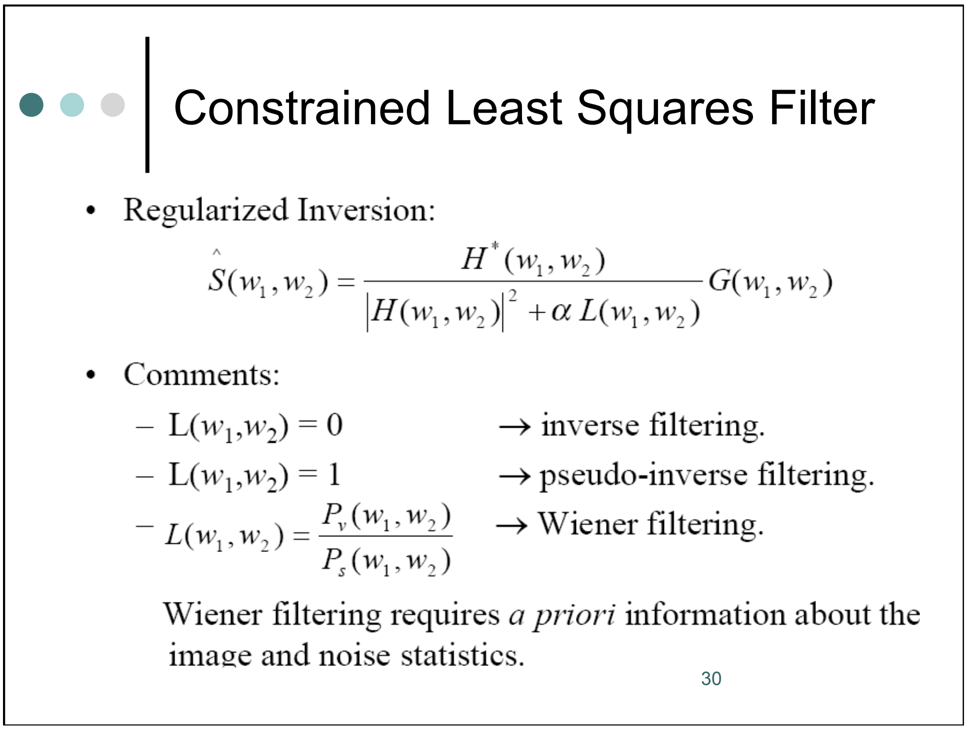
If the operator [**p**] is chosen to be the first derivative, then $|P(\omega)|^2 = \omega^2$, and

$$H_R(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \gamma\omega^2}$$

One may also use the DFT of the first and second derivative kernels as $P(\omega)$

-1	0	1
-2	0	2
-1	0	1

0	-1	0
-1	4	-1
0	-1	0



Constrained Least Squares Filter

- Regularized Inversion:

$$\hat{S}(w_1, w_2) = \frac{H^*(w_1, w_2)}{\left|H(w_1, w_2)\right|^2 + \alpha\, L(w_1, w_2)} G(w_1, w_2)$$

- Comments:
  - $L(w_1,w_2) = 0$ → inverse filtering.
  - $L(w_1,w_2) = 1$ → pseudo-inverse filtering.
  - $L(w_1, w_2) = \frac{P_v(w_1, w_2)}{P_s(w_1, w_2)}$ → Wiener filtering.

Wiener filtering requires *a priori* information about the image and noise statistics.

Wiener vs. Constrained Least Squares Filter

Power Sprectrum Equalization

$$G(w_1,w_2)=H(w_1,w_2)S(w_1,w_2)+V(w_1,w_2)$$

$$\Downarrow$$

$$P_g(w_1,w_2)=|H(w_1,w_2)|^2 P_s(w_1,w_2)+P_v(w_1,w_2)$$

$$\hat{S}(w_1,w_2)=\phi(w_1,w_2)G(w_1,w_2)$$

$$\Downarrow$$

$$P_{\hat{s}}(w_1,w_2)=|\phi(w_1,w_2)|^2 P_g(w_1,w_2)=P_s(w_1,w_2)$$

Power Sprectrum Equalization

$$|\phi(w_1, w_2)| = \frac{1}{\left(|H(w_1, w_2)|^2 + \frac{P_v(w_1, w_2)}{P_s(w_1, w_2)}\right)^{1/2}}$$

This filter may be thought of as being the geometric mean of the inverse filter and Wiener filter and is sometimes referred to as the homomorphic filter.

Note that this formulation only determines the magnitude of the restoration filter.

Homomorphic Filter Behavior

$$|\phi(w_1, w_2)| = \frac{1}{\left(|H(w_1, w_2)|^2 + \frac{P_v(w_1, w_2)}{P_s(w_1, w_2)}\right)^{1/2}}$$

- When noise is small $S_n(\omega) \approx 0$,

$$|\phi(w)| \rightarrow \frac{1}{|H(\omega)|},$$

which is the magnitude of the inverse filter and is similar to the behavior of the Wiener filter.

- When $S_f(\omega) \approx 0$,

$$|\phi(w)| \rightarrow 0,$$

Homomorphic Filter Behavior

$$|\phi(w_1, w_2)| = \frac{1}{\left(|H(w_1, w_2)|^2 + \frac{P_v(w_1, w_2)}{P_s(w_1, w_2)}\right)^{1/2}}$$

- When $H(\omega) \approx 0$,

$$|\phi(w)| \rightarrow \left[\frac{S_f(\omega)}{S_n(\omega)}\right]^{\frac{1}{2}},$$

which is distinctly different from the Wiener filter which would have resulted in a zero estimate.

Maximum *a Posteriori* (MAP) Filter

$$\mathbf{g} = [\mathbf{h}]\mathbf{f} + \mathbf{n}$$

Given the blur matrix $[\mathbf{h}]$, the probability distribution $p_n(\mathbf{n})$ of the noise $\mathbf{n}$, and the probability distribution $p_f(\mathbf{f})$ of the original scene $\mathbf{f}$, find the solution $\hat{\mathbf{f}}$ which maximizes $p(\hat{\mathbf{f}}|\mathbf{g})$, the conditional probability of the object given the recorded image.

The MAP technique allows the incorporation of statistical *a priori* knowledge about the signal and the noise in the restoration process.

MAP Filter

The MAP solution $\hat{\mathbf{f}}$ seeks an original scene which most likely gave rise to the recorded image. The MAP estimate is the solution to the set of equations

$$\frac{\partial}{\partial \hat{\mathbf{f}}} \ln \left\{ p(\mathbf{g}|\hat{\mathbf{f}}) \right\} + \frac{\partial}{\partial \hat{\mathbf{f}}} \ln \left\{ p_f(\hat{\mathbf{f}}) \right\} = 0$$

which are usually nonlinear in $\mathbf{f}$.

MAP Filter Special Case: Gaussian Signal and Noise

For the signal, assume a Gaussian distribution with nonstationary mean $\bar{\mathbf{f}}$ and nonstationary covariance matrix $[\mathbf{K}_f]$. For the noise, assume a zero-mean Gaussian distribution with covariance matrix $[\mathbf{K}_n]$:

$$\boxed{p_f(\mathbf{f}) = a \;\; exp \left\{ -\frac{1}{2}(\mathbf{f} - \bar{\mathbf{f}})^t [\mathbf{K}_f]^{-1} (\mathbf{f} - \bar{\mathbf{f}}) \right\}}$$

$$\boxed{\begin{array}{c} p_n(\mathbf{n}) = b \;\; exp \left\{ -\frac{1}{2}\mathbf{n}^t [\mathbf{K}_n]^{-1} \mathbf{n} \right\} \\ \\ p(\mathbf{g}|\mathbf{f}) = a \;\; exp \left\{ -\frac{1}{2}(\mathbf{g} - [\mathbf{h}]\mathbf{f})^t [\mathbf{K}_n]^{-1} (\mathbf{g} - [\mathbf{h}]\mathbf{f}) \right\} \end{array}}$$

Linear MAP Filter

The MAP solution in this case is a linear filter:

$$[\mathbf{h}]^t[\mathbf{K}_n]^{-1}(\mathbf{g}-[\mathbf{h}]\hat{\mathbf{f}}) \; - \; [\mathbf{K}_f]^{-1}(\hat{\mathbf{f}}-\bar{\mathbf{f}}) = 0$$

$$\hat{\mathrm{f}} = \left([\mathbf{h}]^t[\mathbf{K}_n]^{-1}[\mathbf{h}] + [\mathbf{K}_f]^{-1}\right)^{-1}\left([\mathbf{h}]^t[\mathbf{K}_n]^{-1}\mathbf{g} + [\mathbf{K}_f]^{-1}\bar{\mathbf{f}}\right)$$

which is of the form

$$\hat{\mathbf{f}} = [\mathbf{w}]\mathbf{g} + \mathbf{b}$$

Linear MAP Filter

where

$$W(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \frac{S_n(\omega)}{S_f(\omega)}}$$

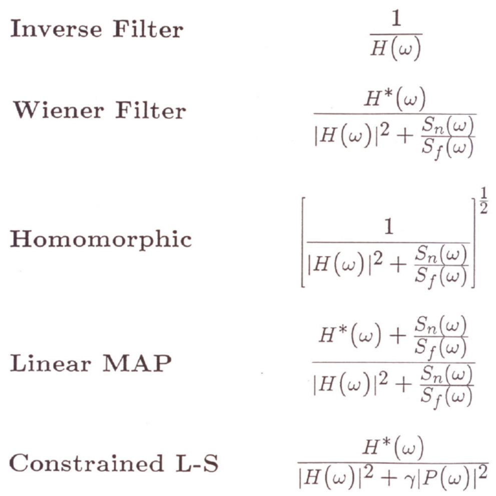
and

$$B(\omega) = \frac{\frac{S_n(\omega)}{S_f(\omega)}}{|H(\omega)|^2 + \frac{S_n(\omega)}{S_f(\omega)}}\overline{F}(\omega)$$

Under the assumption that $\overline{F}(\omega) = G(\omega)$, and $\frac{S_n(\omega)}{S_f(\omega)} = \gamma$, we get

$$H_R(\omega) = \frac{H^*(\omega) + \gamma}{|H(\omega)|^2 + \gamma}$$

Summary of Linear Filters



Inverse Filter	$\frac{1}{H(\omega)}$
Wiener Filter	$\frac{H^*(\omega)}{\|H(\omega)\|^2+\frac{S_n(\omega)}{S_f(\omega)}}$
Homomorphic	$\left[\frac{1}{\|H(\omega)\|^2+\frac{S_n(\omega)}{S_f(\omega)}}\right]^{\frac{1}{2}}$
Linear MAP	$\frac{H^*(\omega)+\frac{S_n(\omega)}{S_f(\omega)}}{\|H(\omega)\|^2+\frac{S_n(\omega)}{S_f(\omega)}}$
Constrained L-S	$\frac{H^*(\omega)}{\|H(\omega)\|^2+\gamma\|P(\omega)\|^2}$

Project 3.2b

Image Restoration

Due 29.12.2013 Sunday

Problem 3.2 Image Restoration with Wiener Filter

1. Select a 512x512 monochrome image and display it.
2. Blur the image with a uniform 16x16 blur. Display the blurred image.
3. Add 20 dB Gaussian noise to the blurred image and display the result.
4. Divide the image into 64x64 regions, apply Hanning window to all regions. Display the image with Hanning windows applied.
5. Calculate the 2-D DFT of all regions and compute the 2-D power spectrum of the observed image as

$$P_g(k,l) = \frac{1}{64}\sum_{ij}\left|DFT_{64x64}\{g_{ij}\}\right|^2$$

6. Compute the variance of the noise using nearly uniform regions (calculate the variance in each block, then average these variances).
7. Compare it with the actual value given in Step 3.

Problem 3.2 (cont.)

1. Calculate the frequency response of the Wiener filter as (use the given noise variance)

$$\phi(k,l) = \frac{H^*(k,l)}{|H(k,l)|^2 + \dfrac{\sigma_v^2}{P_s(k,l)}} \qquad P_s(k,l) = \max\left(P_g(k,l) - \sigma_v^2, 0\right)$$

2. Plot the above filter frequency response and comment on its characteristics.
3. Obtain the filter impulse response by computing the 64x64 inverse DFT of the fiter frequency response. Plot the impulse response and comment on its shape.
4. Convolve the 512x512 blurred and noisy image with the Wiener filter impulse response to obtain the restored image. Display the restored image and comment on its characteristics.
5. Compare your result with MATLAB's Wiener filter.

Image Restoration with POCS

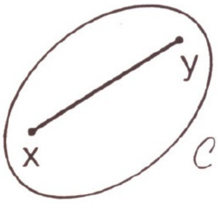
Convex Set

- A closed convex set $\mathcal{C}$ in $\mathcal{H}$

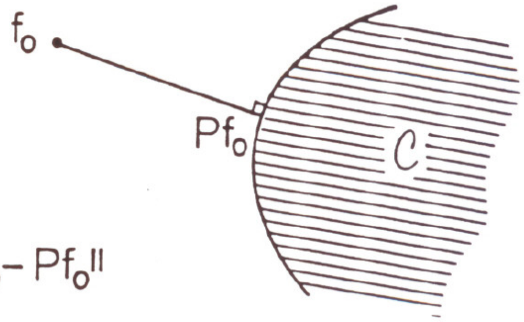
  $\mathcal{C}$ is convex iff for any x and y in $\mathcal{C}$, z defined by

  $$z \triangleq \mu x + (1-\mu) y \qquad (0 \leq \mu \leq 1)$$

  is also in $\mathcal{C}$.



Projection onto a Convex Set



$$\min_{y\in\mathcal{C}} \|f_0-y\| = \|f_0-Pf_0\|$$

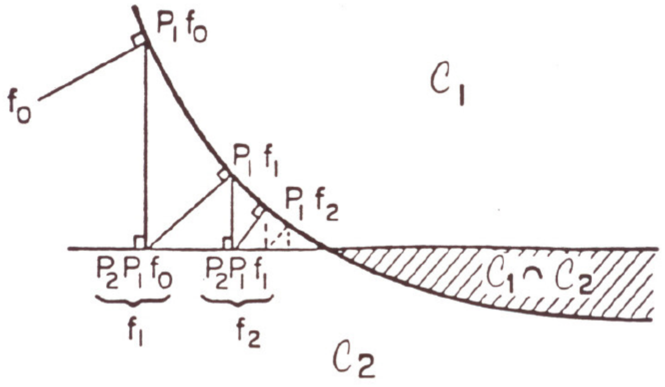
Convergence

To find a point in the intersection of m closed convex sets $\mathcal{C}_1$, $\mathcal{C}_2, \ldots,$ $\mathcal{C}_m$ in a Hilbert space start with an arbitrary initialization function f_0 and perform successive projections onto convex sets. The sequence $\{f_k\}$ generated by

$$P_m\, P_{m-1}\, \cdots P_1\, f_k = f_{k+1}$$

converges weakly to a point of $\mathcal{C}_0 = \bigcap_{i=1}^{m} \mathcal{C}_i$

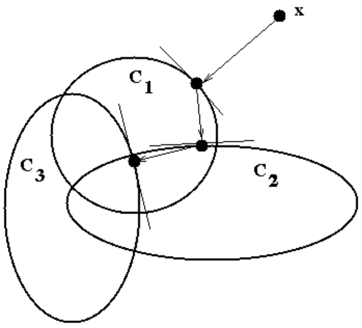
Successive Projections



$$f_{k+1} = P_2 P_1 f_k \quad , \quad k = 0, 1 \ldots$$

The Method of POCS

- Define closed, convex constraint sets, *C*
- A *projection* operator **P**, maps an arbitrary point to the closest point in *C*.
- Solution set is intersection of all constraint sets



- Start with an arbitrary initial estimate, then project onto all constraint sets iteratively
- Relaxed projection operators, $\mathbf{T} = (1-\lambda)\,\mathbf{I} + \lambda\,\mathbf{P}; \quad 0 < \lambda < 2$

POCS Example:

Solving a Linear System of Equations

$$\underline{a}_i^T \underline{x} = b_i, \qquad i = 1,\ldots,M$$

$\underline{x}$ is an $M \times 1$ vector of unknowns

$b_i, \quad i = 1,\ldots,M$ are scalars

$\underline{a}_i, \quad i = 1,\ldots,M$ are $M \times 1$ coefficient vectors

$$\begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1M} \\ \vdots & & & \\ \vdots & & & \\ \mathrm{a}_{M1} & \mathrm{a}_{M2} & \cdots & \mathrm{a}_{MM} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix}$$

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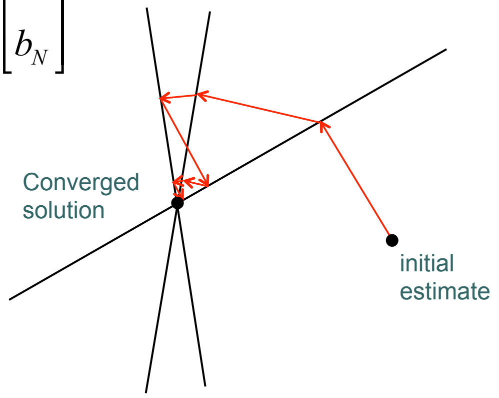
Solution of a Set of Linear Equations

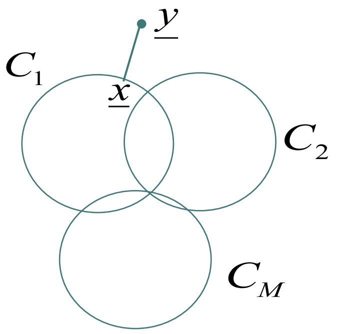
$$\begin{bmatrix} \overline{a}_1^T \\ \vdots \\ \overline{a}_N^T \end{bmatrix} \overline{x} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix} \longleftarrow$$

Each row represents a constraint



Solution via POCS

$$C_i = \{\underline{x} | \underline{a}_i^T \underline{x} = b_i\}, \qquad i = 1, \ldots, M$$



POCS: Example (cont’ d)

Projection of an arbitrary $M \times 1$ vector $\underline{y}$ onto C_i

Minimize $\|\underline{y}\text{-}\underline{x}\|^2$ under the constraint that $\underline{a}_i^T \underline{x} = b_i$

Thus, we need to find $\underline{x}$ that minimizes

$$\|\underline{y}\text{-}\underline{x}\|^2 + \lambda\left(\underline{a}_i^T \underline{x} - b_i\right) = E(\underline{x}, \lambda)$$

then we say that $\underline{x}$ is the projection of $\underline{y}$ onto set C_i

$$\frac{\partial}{\partial \underline{x}} E(\underline{x}, \lambda) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} E(\underline{x}, \lambda) = 0$$

POCS: Example (cont’d)

$$\frac{\partial}{\partial \underline{x}} E(\underline{x},\lambda) = 0 \quad \Rightarrow -2(\underline{y}-\underline{x}) + \lambda \underline{a}_i = 0$$

$$(1) \Rightarrow \underline{x} = \underline{y} - \frac{\lambda}{2}\underline{a}_i$$

Substitute this into the constraint equation

$$\underline{a}_i^T\left(\underline{y}-\frac{\lambda}{2}\underline{a}_i\right) = \underline{b}_i \Rightarrow \underline{a}_i^T\underline{y} - \frac{\lambda}{2}\underline{a}_i^T\underline{a}_i = \underline{b}_i$$

$$\lambda = 2\frac{\underline{a}_i^T\underline{y}-\underline{b}_i}{\|\underline{a}_i\|^2} \quad \text{put this into (1)}$$

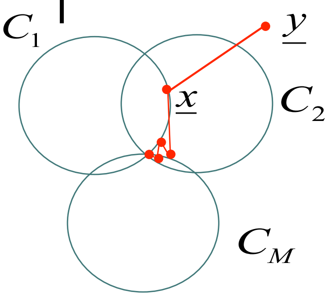
and finally

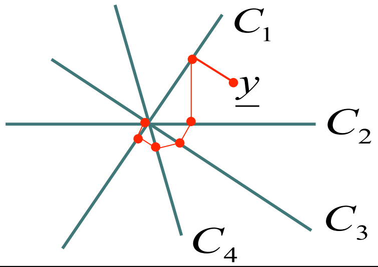
$ax - b = 0$

$$\underline{x} = \boxed{\underline{y} - \frac{\underline{a}_i^T\underline{y}-\underline{b}_i}{\|\underline{a}_i\|^2}\underline{a}_i}$$

Projection of $\underline{y}$ onto C_i

POCS: Example (cont’d)





Define projection operator P_i such that

$$P_i\underline{y} = \underline{y} - \frac{\underline{a}_i^T\underline{y}-\underline{b}_i}{\|\underline{a}_i\|^2}\mathrm{a}_i$$

Algorithm

$$\underline{x} = (P_M \dots P_2P_1)(P_M \dots P_2P_1)\underline{y}$$

Convex Sets for Restoration

$$\mathcal{C}_v = \left\{ \hat{\mathbf{f}} \middle| \ \|\mathbf{g} - [\mathbf{h}]\hat{\mathbf{f}}\|^2 \leq \delta_v \right\}$$

$$\mathcal{C}_m = \left\{ \hat{\mathbf{f}} \middle| \ \left| \sum_i (g_i - [\mathbf{h}\hat{\mathbf{f}}]_i) \right| \leq \delta_m \right\}$$

$$\mathcal{C}_o = \left\{ \hat{\mathbf{f}} \middle| \ |g_i - [\mathbf{h}\hat{\mathbf{f}}]_i| \leq \delta_o \right\}$$

Convex Sets for Restoration

$$\mathcal{C}_p = \left\{ \hat{\mathbf{f}} \middle| \ |G(k) - H(k)F(k)|^2 \leq \delta_p \right\}$$

$$\mathcal{C}_n = \left\{ \hat{\mathbf{f}} \middle| \ f_i \geq 0 \right\}$$

The quantity $\mathbf{r} = \mathbf{g} - [\mathbf{h}]\hat{\mathbf{f}}$ is called the residual.

Image Restoration Using POCS

- Each *a priori* information and constraint restrict the function f to a closed convex set in a Hilbert space.

- Thus, for m pieces of information there are m closed convex sets C_i, $i = 1, 2, ..., m$ and
$f \in C_0 = \bigcap_{i=1}^{m} C_i$

- Given C_i and their projectors P_i,

$$f_{k+1} = P_m \; P_{m-1} \cdots P_1 \; f_k$$

converges to a solution in the intersection C_0.

Next Lecture

- IMAGE RECONSTRUCTION