Solution of nonlinear algebraic equation

Nonlinear algebraic equations are defined as those which contain powers of variables and/or transcendental functions. Such equations arise frequently in engineering, especially when one is dealing with optimization, differential equations, the Eigen problems.

1- Bisection Method

The bisection method is used to find the roots of a polynomial equation. It separates the interval and subdivides the interval in which the root of the equation lies. The principle behind this method is the intermediate theorem for continuous functions. It works by narrowing the gap between the positive and negative intervals until it closes in on the correct answer. This method narrows the gap by taking the average of the positive and negative intervals. It is a simple method and it is relatively slow. The bisection method is also known as interval halving method, root-finding method, binary search method or dichotomy method.

Let us consider a continuous function "f" which is defined on the closed interval [a, b], is given with f(a) and f(b) of different signs. Then by intermediate theorem, there exists a point x belong to (a, b) for which f(x) = 0.

Bisection method algorithm

Follow the below procedure to get the solution for the continuous function:

For any continuous function f(x),

Bisection method Steps (Rule)

Step-1:	Find points a and b such that $a < b$ and $f(a) \cdot f(b) < 0$.
Step-2:	Take the interval $[a, b]$ and find next value $c = \frac{a+b}{2}$
Step-3:	If $f(c) = 0$ then x_0 is an exact root, else if $f(a) \cdot f(c) < 0$ then $b = c$, else if $f(c) \cdot f(b) < 0$ then $a = c$.
Step-4:	Repeat steps 2 & 3 until $f(c) = 0$ or $\lvert f(c) \rvert \le$ Accuracy , Accuracy=0.01

Example: Determine the root of the given equation x^2-3 = 0 for x ∈ [1, 2]

Solution:

Given: $x^2-3 = 0$

Let $f(x) = x^2-3$

Now, find the value of f(x) at a= 1 and b=2.

$f(x=1) = 1^2-3 = 1 – 3 = -2 < 0$

$f(x=2) = 2^2-3 = 4 – 3 = 1 > 0$

The given function is continuous, and the root lies in the interval [1, 2].

Let “c” be the midpoint of the interval.

I.e., c = (1+2)/2

c =3 / 2

c = 1.5

Therefore, the value of the function at “c” is

$f(c) = f(1.5) = (1.5)^2-3 = 2.25 – 3 = -0.75 < 0$

If *f(c)* < 0, assume a = c. and

If *f(c)* > 0, assume b = c.

f(c) is negative, so a is replaced with c = 1.5 for the next iterations.

The iterations for the given functions are:

Iterations	a	b	f(a)	f(b)	c	f(c)	b-a
1	1	2	-2	1	1.5	-0.75	1
2	1.5	2	-0.75	1	1.75	0.062	0.5
3	1.5	1.75	-0.75	0.0625	1.625	-0.359	0.25
4	1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	0.125
5	1.6875	1.75	-01523	0.0625	1.7188	-0.0457	0.0625
6	1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	0.0312
7	1.7188	1.7344	-0.0457	0.0081	1.7266	-0.0189	0.0156

So, at the seventh iteration, we get the final interval [1.7266, 1.7344]

Hence, 1.7344 is the approximated solution.

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Example: Find a root of an equation $f(x) = x^3 - x - 1$ using Bisection method
Solution:
$f(0) = 0^3 - 0 - 1 = -1$
$f(1) = 1^3 - 1 - 1 = -1 < 0$
$f(2) = 2^3 - 2 - 1 = 5 > 0$

[1,2] a=1 , b=2

Approximate root of the equation *x*3-*x*-1=0 using Bisection method is 1.32471

n	*a*	*f(a)*	*b*	*f(b)*	*c*	*f(c)*	b-a
1	1	-1	2	5	1.5	0.875	1
2	1	-1	1.5	0.875	1.25	-0.29688	0.5
3	1.25	-0.29688	1.5	0.875	1.375	0.22461	0.25
4	1.25	-0.29688	1.375	0.22461	1.3125	-0.05151	0.125
5	1.3125	-0.05151	1.375	0.22461	1.34375	0.08261	0.0625
6	1.3125	-0.05151	1.34375	0.08261	1.32812	0.01458	0.03125
7	1.3125	-0.05151	1.32812	0.01458	1.32031	-0.01871	0.0156
8	1.32031	-0.01871	1.32812	0.01458	1.32422	-0.00213	0.00781
9	1.32422	-0.00213	1.32812	0.01458	1.32617	0.00621	0.0039
10	1.32422	-0.00213	1.32617	0.00621	1.3252	0.00204	0.00195
11	1.32422	-0.00213	1.3252	0.00204	1.32471	-0.00005	0.00098

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Example: Find a root of an equation $f(x) = 2x^3 - 2x - 5$ using Bisection method
Solution:
Here $2x^3 - 2x - 5 = 0$

Let $f(x) = 2x^3 - 2x - 5$
Here

x	0	1	2

$f(x)$	-5	-5	7

1*st* iteration :

Here $f(1)=-5<0$ and $f(2)=7>0$

∴ Now, Root lies between 1 and 2

$x0=1+22=1.5$

$f(x0)=f(1.5)=2\cdot1.53-2\cdot1.5-5=-1.25<0$

Approximate root of the equation $2x3-2x-5=0$ using Bisection mehtod is 1.60059

n	*a*	*f*(*a*)	*b*	*f*(*b*)	*c*=*a*+*b*/2	*f*(*c*)	b-c
1	1	-5	2	7	1.5	-1.25	1
2	1.5	-1.25	2	7	1.75	2.21875	0.5
3	1.5	-1.25	1.75	2.21875	1.625	0.33203	0.25
4	1.5	-1.25	1.625	0.33203	1.5625	-0.49561	0.125
5	1.5625	-0.49561	1.625	0.33203	1.59375	-0.09113	0.0625
6	1.59375	-0.09113	1.625	0.33203	1.60938	0.1181	0.03125
7	1.59375	-0.09113	1.60938	0.1181	1.60156	0.0129	0.01563
8	1.59375	-0.09113	1.60156	0.0129	1.59766	-0.03926	0.00781
9	1.59766	-0.03926	1.60156	0.0129	1.59961	-0.01322	0.0039
10	1.59961	-0.01322	1.60156	0.0129	1.60059	-0.00017	0.00195

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Example- Find a root of an equation $f(x) = \sqrt{12}$ using Bisection method

Solution:

Here $\sqrt{12}=0$

Let $f(x=\sqrt{12}$

$x=\sqrt{12}$
$\therefore x^2=12$
$\therefore x^2-12=0$
Here

x	0	1	2	3	4
$f(x)$	-12	-11	-8	-3	4

1*st* iteration :
Here $f(3)=-3<0$ and $f(4)=4>0$
∴ Now, Root lies between 3 and 4
Approximate root of the equation *x*2-12=0 using Bisection method is 3.4641 (After 12 iterations)

n	*a*	*f(a)*	*b*	*f(b)*	*c=a+b2*	*f(c)*	b-c
1	3	-3	4	4	3.5	0.25	1
2	3	-3	3.5	0.25	3.25	-1.4375	0.5
3	3.25	-1.4375	3.5	0.25	3.375	-0.6094	0.25
4	3.375	-0.6094	3.5	0.25	3.4375	-0.1836	0.125
5	3.4375	-0.1836	3.5	0.25	3.4688	0.0322	0.0625
6	3.4375	-0.1836	3.4688	0.0322	3.4531	-0.0759	0.0313
7	3.4531	-0.0759	3.4688	0.0322	3.4609	-0.0219	0.0157
8	3.4609	-0.0219	3.4688	0.0322	3.4648	0.0051	0.0079
9	3.4609	-0.0219	3.4648	0.0051	3.4629	-0.0084	0.0039
10	3.4629	-0.0084	3.4648	0.0051	3.4639	-0.0016	0.0019
11	3.4639	-0.0016	3.4648	0.0051	3.4644	0.0018	0.0009
12	3.4639	-0.0016	3.4644	0.0018	3.4641	0.0001	0.0005

Example: find the root of the following function using Bisection method: $f(x) = x^3 + 3x - 5$

Solution:

$f(0) = 0^3 + 3 * 0 - 5 = -5 < 0$

$f(1) = 1^3 + 3 * 1 - 5 = -1 < 0$

$f(2) = 2^3 + 3 * 2 - 5 = 9 > 0$

[1,2] a=1 , b=2 we will stop when(b-a) <0.01

iteration	a	b	f(a)	f(b)	$c = \frac{a+b}{2}$	f(c)	b-a
1	1	2	-1	9	1.5	2.875	1
2	1	1.5	-1	2.875	1.25	0.703	0.5
3	1	1.25	-1	0.703	1.125	-0.201	0.75
4	1.125	1.25	-0.201	0.703	1.1875	0.237	0.125
5	1.125	1.1875	-0.201	0.237	1.15625	0.014	0.0623
6	1.125	1.15625	-0.201	0.014	1.1406	-0.094	0.03125
7	1.1406	1.15625	-0.094	0.014	1.1484	-0.040	0.0156
8	1.1484	1.15625	-0.040	0.014	1.1523	-----	0.0075

(b-a) at b=1.15625 is < 0.01

The approximate root of the function given is 1.15625

2- Newton-Raphson method

The Newton-Raphson method is one of the most widely used methods for root finding. It can be easily generalized to the problem of finding solutions of a system of non-linear equations, which is referred to as Newton's technique.

Unlike the bisection and false position methods, the Newton-Raphson (N-R) technique requires only one initial value x_0, which we will refer to as the *initial guess* for the root. To see how the N-R method works, we can rewrite the function *f*(*x*) using a Taylor series expansion in (x-x_0):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots = 0$$

Where $f'(x)$ denotes the first derivative of *f*(*x*) with respect to *x*, f'' is the second derivative, and so forth. Now, suppose the initial guess is pretty close to the real root. Then (x-x_0) is small, and only the first few terms in the series are important to get an accurate estimate of the true root, given x_0. By truncating the series at the second term (linear in x), we obtain the N-R iteration formula for getting a better estimate of the true root:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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Example: find the root of the following function using Newton-Raphson method: $f(x) = x^3 - 4x - 9$

Solution:-x=?

$f(0) = 0^3 - 4 * 0 - 9 = -9$

$f(1) = 1^3 - 4 * 1 - 9 = -12$

$f(2) = 2^3 - 4 * 2 - 9 = -9 \quad (-ve)$

$f(3) = 3^3 - 4 * 3 - 9 = 6 \quad (+ve)$

[2, 3] we need to do the initial approximation, for that we need to check which value close to zero

6 is line close to 0, initial approximation= $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_0 = 3$

$f(x_0) = x^3 - 4x - 9$

$f'(x_0) = 3x^2 - 4$

$$x_1 = 3 - \frac{3^3 - 4 * 3 - 9}{3 * 3^2 - 4} = 2.739$$

$$x_2 = 2.739 - \frac{2.739^3 - 4 * 2.739 - 9}{3 * 2.739^2 - 4} = 2.706$$

$$x_3 = 2.706 - \frac{2.706^3 - 4 * 2.706 - 9}{3 * 22.706^2 - 4} = 2.706$$

X=2.706

Example: given that $f(x) = x^3 + 2x - 2$ has a root between 0 &1 , and the root to 2 decimal placing using Newton-Raphson method.

Solution: $f(x) = x^3 + 2x - 2$

$f'(x_0) = 3x^2 + 2$

$f(0) = 0^3 + 2 * 0 - 2 = -2\ (-ve)$

$f(1) = x = 1^3 + 2 * 1 - 2 = 1(+ve)$

1 is line close to 0, initial approximation= $x_0 = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{1^3 + 2 * 1 - 2}{3 * 1^2 + 2} = 1 - \frac{1}{5} = 0.8$$

$$x_2 = 0.8 - \frac{0.8^3 + 2 * (0.8) - 2}{3 * (0.8)^2 + 2} = 0.77149$$

$$x_3 = 0.7714 - \frac{0.7714^3 + 2 * (0.7714) - 2}{3 * (0.7714)^2 + 2} = 0.7713$$

$x_4 = 0.7713$,
$x_5 = 0.7713$, $x = 0.77$