Runge-Kutta Methods

Runge- Kutta 2nd order

For the differential equation

$$\bar{y} = \frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$x = x_0 + h$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = h\, f(x_n + h, y_n + k_1)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$\bar{y} = \frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$x = x_0 + h$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + k_1)$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

Example: apply Runge- kutta method of second order to find an approximate value of y given that $\frac{dy}{dx} = x^2 + y$ and y(0)=1, h=0.01, y(0.01)=y_1.

Solution:- x_0=0, y_0=1, $f(x,y) = x^2 + y$

$x_1 = x_0 + h, \quad x_1 = 0 + 0.01, \quad x_1 = 0.01$

$k_1 = hf(x_n, y_n) = (0.01)f(0,1) = (0.01)(0^2 + 1) = 0.01$

$$k_2 = h\, f(x_n + h, y_n + k_1) = (0.01)f(0 + 0.01, 1 + 0.01)$$
$$= (0.01)(0.01, 1.01) = (0.01)(0.01^2 + 1.01)$$
$$= (0.01)(1.0101) = 0.010101$$

$y_1 = y(0.01) = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.01 + 0.010101) =$
1.0100505

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Runge- Kutta 3rd order

For the differential equation

$$\bar{y} = \frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$
$$x = x_0 + h$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f(x_n + h, y_n + 2k_2 - k_1)$$
$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$\bar{y} = \frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$
$$x = x_0 + h$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = f(x_n + h, y_n + 2k_2 - k_1)$$
$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3)$$

Example:- use Runge kutta 3rd

order method to find y when x=0.1 ,y=1 at x=0, $\frac{dy}{dx} = 3x + y^2$

Solution:

$k_1 = hf(x_n, y_n) = 0.1f(0,1) = 0.1$

$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = 0.1 * f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1 * f(0.05, 1.05) =$

0.12525

$k_3 = h\,f(x_n + h, y_n + 2k_2 - k_1) = 0.1 * f(0 + 0.1, 1 + 2 * 0.12525 -$

$0.1 = 0.1 * f(0.1, 2.1505) = 1.623$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3) = 1 + \frac{1}{6}(0.1 + 4 * 0.125 + 1.623) =$

1.3705

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Runge- Kutta 4th order

For the differential equation

$$\bar{y} = \frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$x = x_0 + h$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

$$\bar{y} = \frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$x = x_0 + h$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{h}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

Example:- **use Runge kutta 4th order method to find y(0.1) when h=0.1, y(0)=2,** $\boldsymbol{\frac{dy}{dx} = y - x}$

Solution:

$k_1 = hf(x_n, y_n) = 0.1f(0,2) = 0.1 * (2 - 0) = 0.1 * (2) = \mathbf{0.2}$

$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) = 0.1f\left(0 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right) = 0.1f(0.05,2.1) = 0.1(2.1 - 0.05,) = \mathbf{0.205}$

$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) = 0.1\,f\left(0 + \frac{0.1}{2}, 2 + \frac{0.205}{2}\right) = 0.1\,f(0.05,2.1025\,)$

$= 0.1\,(2.1025 - 0.05) = \mathbf{0.20525}$

$k_4 = h\,f(x_n + h, y_n + k_3) = 0.1\,f(0 + 0.1, 2 + 0.20525) = 0.1\,f(0.1,2.20525)$

$= \mathbf{0.210525}$

$y_{(0.1)} = y_n + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4] = 2 + \frac{1}{6}[\mathbf{0.2} + 2(\mathbf{0.205} + \mathbf{0.210525}) + \mathbf{0.210525}] = 2.20693$

Example: If $\bar{y} = x - 0.2\,y$, $y(0) = 1$, find y at $x = 1$ using Runge-Kutta 2nd order.

Sol:

$h = x - x_0 \rightarrow \; h = 1 - 0 \; \rightarrow h = 1$

$k_1 = hf(x_n, y_n) = 1f(0,1) = 0 - 0.2 * 1 = \; \mathbf{-0.2}$

$k_2 = h\, f(x_n + h, y_n + k_1) = 1\, f(0 + 1, 1 - \mathbf{0.2}) = f(1, \mathbf{0.8}) = \mathbf{0.84}$

$y_{n+1} = 1 + \frac{1}{2}(\mathbf{-0.2} + \mathbf{0.84}) = 1.32$

$Example$: Repeat the example by using Runge- Kutta 3rd order.

$k_1 = f(0,1) \; \rightarrow \;\; k_1 = 0 - 0.2 * 1 = \; -0.2$

$k_2 = \; f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) \rightarrow \;\; k_2 = \left(1 + \frac{1}{2}\right) - 0.2 * \left(1 + \; \frac{1}{2} * -0.2\right)$

= 1.32

$k_3 = \; f(x_0 + h, y_0 + h(2k_2 - k_1) \; \rightarrow \;\; k_2 = (1 + 1\,) - 0.2 *$

$(1 + \; 1(2 * 1.32 + 0.2)$

= 1.232

and this leads to $y = y_0 + \frac{h}{6}(k_1 + \; 4k_2 + k_3)$

$\rightarrow y = 1 + \frac{1}{6}(-0.2 + \; 4 * 1.32 + 1.232)$

= 2.052

$EX_3/$ If $\bar{y} = e^{0.8x} + \; y$ & $y(0) = \; 2$, find y at $x = 2$ using Runge- Kutta 4th order.

Sol:

$k_1 = f(x_0, y_0) \;\; \rightarrow \;\; k_1 = 1 + 2$

= 3

$k_2 = \; f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) \rightarrow \;\; k_2 = (e^{0.8} + 2 + 1 * 3\,)$

= 7.226

$k_3 = \; f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2\right) \rightarrow \;\; k_3 = (e^{0.8} + 2 + 1 * 7.226)$

= 11.4511

$k_4 = f(x_0 + h, y_0 + hk_3) \rightarrow k_4 = (e^{0.8} + 2 + 2 * 11.4511 \ )$
=27.1277

$y = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 2 + \frac{2}{6}(3 + 2 * 7.226 + 2 * 11.4511 + 27.1277)$

=24.494

EX_4/ Use Runge-Kutta method (4^{th} order) to find $y(0.3)$& $z(0.3)$ for the system.

$\bar{y} = y + 2e^{-x}$

$\bar{z} = z^2 - y$

$y(0) = 1$ & $z(0) = 2$

Sol:

$x_0 = 0 , y_0 = 1, z_0 = 2$ & $h = 0.3$

$k_1 = f(x_0, y_0, z_0) \rightarrow k_1 = 1{+}2 * e^0$

$= 3$

$n_1 = g(x_0, y_0, z_0) \rightarrow k_1 = 4 - 1$

$= 3$

$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1, z_0 + \frac{h}{2}n_1\right) \rightarrow k_2 = \left(1 + \frac{0.3}{2} * 3 + 2 * e^{-1.15}\right)$

$= 2.0832$

$n_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1, z_0 + \frac{h}{2}n_1\right) \rightarrow n_2 = \left((2 + \frac{0.3}{2} * 3)^2 - (1 + \frac{0.3}{2} * 3\right)$

$= 4.5525$

$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2, z_0 + \frac{h}{2}n_2\right) \rightarrow k_3 = \left(1 + \frac{0.3}{2} * 2.0832 + 2 * e^{-1.15}\right)$

$= 1.9458$

$n_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2, z_0 + \frac{h}{2}n_2\right) \rightarrow \ n_3 = \left((2 + \frac{0.3}{2} * 4.5525)^2 - (1 + \frac{0.3}{2} * 2.0832)\right)$

$= 5.8853$

$k_4 = f(x_0 + h, y_0 + hk_3, z_0 + hn_3) \rightarrow \ k_4 = (1 + 0.3 * 1.9458 + 2 * e^{-1.3})$

$=2.129$

$n_4 = f(x_0 + h, y_0 + hk_3, z_0 + hn_3) \rightarrow \ n_4 = ((2 + 0.3 * 5.8853)^2 - (1 + 0.3 * 1.9458))$

$= 12.596$

$y = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 2 + \frac{0.3}{6}(3 + 2 * 2.0832 + 2 * 1.9458 + 2.129)$

$=1.5552$

$z = z_0 + \frac{h}{6}(n_1 + 2n_2 + 2n_3 + n_4)$

$= 2 + \frac{0.3}{6}(3 + 2 * 4.5525 + 2 * 5.8853 + 12.596) \rightarrow z = 3.82258$

EX_5/ Use Runge-Kutta method (4^{th} order) to find $y(1)$& $\bar{y}(1)$ for

$\bar{\bar{y}} + 2\bar{y} + y = 0$ With $y(0) = 2$ & $\bar{y}(0) = 0$

Sol:

Let $\bar{y} = z \ \rightarrow \ \bar{z} + 2z + y = 0 \ \therefore \ \bar{z} = -2z - y$ now we have two equations

$\bar{y} = z$

$\bar{z} = -2z - y$

And $h = 1, y(0) = 2$ & $z(0) = 0$

$k_1 = f(0,2,0) \ \rightarrow \ k_1 = 0$

$n_1 = g(0,2,0) \ \rightarrow \ n_1 = -2 * 0 - 2$

$= \ -2$

$k_2 = f\left(0+\frac{1}{2}, 2+\frac{1}{2}*0, 0+\frac{1}{2}*-2\right) \rightarrow \; k_2 = -1$

$n_2 = f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}k_1, z_0+\frac{h}{2}n_1\right) \rightarrow \; n_2 = \left(-2*\frac{1}{2}*-2 \; - (2+\frac{1}{2}*\right.$

$\left.0\right)$

$= -4$

$k_3 = f\left(0+\frac{1}{2}, 2+\frac{1}{2}*-1, 0+\frac{1}{2}*-4\right) \rightarrow \; k_3 = \left(0+\frac{1}{2}*-1\right)$

$= \; -0.5$

$n_3 = f\left(0+\frac{1}{2}, 2+\frac{1}{2}*-1, 0+\frac{1}{2}*-4\right) \rightarrow \; k_3 = \left(-2*\frac{1}{2}*-4 \; - (2+\right.$

$\left.\frac{1}{2}*-1\right)$

= 2.5

$k_4 = \; f(0+1, 2+1*-0.5, 0+1*2.5) \rightarrow$

$k_4 = 2.5$

$n_4 = \; f(0+1, 2+1*-0.5, 0+1*2.5) \rightarrow \; \rightarrow \; n_4 = (-2*2.5*$

$-(2+1*-0.5) = -6.5$

$y = 2+\frac{1}{6}(0+(2*-1)+(2*-0.5)+2.5)$

=1.91667

$y = 0+\frac{1}{6}(-2+(2*-4)+(2*-2.5)-6.5)$

= 3.58333

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Adams-Bash forth Predictor- Corrector Method

$\bar{y} = \frac{dy}{dx} = f(x, y)$

$y(x_0) = \; y_0$

$x = x_0 + h$

To Find: $y(x_n)$ ***when n=4***

Step1: if $y(x_1), y(x_2), y(x_3)$ is given that's ok , if $y(x_1), y(x_2), y(x_3)$ is not given then we find $y(x_1), y(x_2), y(x_3)$ using any one of the methods like Taylor's series method, Euler's method, Modified Euler's method or Runge Kutta method

Step2: calculate

$\boldsymbol{f_0}= \boldsymbol{f(x_0, y_0)}$, $\boldsymbol{f_1}= \boldsymbol{f(x_1, y_1)}$,

$\boldsymbol{f_2}= \boldsymbol{f(x_2, y_2)}$, $\boldsymbol{f_3}= \boldsymbol{f(x_3, y_3)}$

Step3: By Adams-Bash forth predictor formula

$$y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0)$$

Step4: then find $\boldsymbol{f_4}= \boldsymbol{f(x_4, y_4)}$, when $\boldsymbol{x_4 = x_3 + h}$

Step5: By Adams-Bash forth corrector formula

$$y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1)$$

We repeat this step until y_4 unchanged

Example: Using Adams-Bash forth method ,obtain the solution of $\frac{dy}{dx} = x - y^2$ at x=0.8 given that

x	0	0.2	0.4	0.6
y	0	0.02	0.079	0.1762

Solution: $f(x,y) = x - y^2$, h=0.2

$x_0 = 0$, $y_0 = 0$, $f_0 = f(x_0, y_0) = 0$

$x_1 = 0.2$, $y_1 = 0.02$, $f_1 = f(x_1, y_1) = 0.1996$

$x_2 = 0.4$, $y_2 = 0.0795$, $f_2 = f(x_2, y_2) = 0.3937$

$x_3 = 0.6$, $y_3 = 0.1762$, $f_3 = f(x_3, y_3) = 0.5690$

Using Adams-Bash forth predictor formula

$$y_4 = y_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0)$$

$$y_4 = 0.1762 + \frac{0.2}{24}\big(55 * (0.5690) - 59 * (0.3937) + 37 * (0.1996) - 9(0)\big) = 0.3046$$

$f_4 = f(x_4, y_4)$=0.8-$(0.5739)^2$ =0.7072

Using Adams-Bash forth corrector formula

$y_4 = y_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) = 0.1762 + \frac{0.2}{24}(9 * (0.3046\ ) + 19 * (0.5690) - 5 * (0.3937) + 0.1996) = 0.3046$

$y_4 = y(0.8) = 0.3046$

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Homework1: by using Runge- Kutta method of second order, solve $y' = x^2 - 2xy$, y(1)=0, h=0.2, find one step

Homework2: by using Runge- Kutta method of third order, solve $y' = x + y^2$, y(0)=1, h=0.1, find y(0.1)?