Example(3.1)

Estimate the natural logarithm of 2 using linear interpolation. First, perform the computation by interpolating between ln (1) = 0 and ln (6) = 1.791759. Then, repeat the procedure, but use a smaller interval from ln (1) to ln (4) = 1.386294. Note that the true value of ln (2) is 0.6931472.

Solution

We use a linear interpolation for (2) from x_0 = 1 to x_1 = 6 to give

$$y_1(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0)$$

$$y_1(2) = 0 + \frac{1.791759 - 0}{6 - 1}(2 - 1) = 0.3583519$$

Using a smaller interval from x_0 = 1 to x_1 = 4

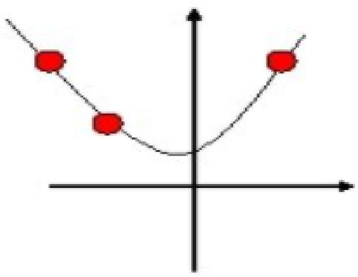
$$y_1(2) = 0 + \frac{1.386294 - 0}{4 - 1}(2 - 1) = 0.4620981$$

****

B- Quadratic Interpolation

a strategy for improving the estimate is to introduce some curvature into the line connecting the points. This can be accomplished with a second order polynomial. If three data points are available, this can be accomplished with a second-order polynomial (also called a quadratic polynomial or a parabola).

A simple procedure can be used to determine the values of the coefficients



$$(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \ldots\ldots\ldots\ldots (2)$$

$$f(x_1,x_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1}\ldots\ldots\ldots\ldots\ldots(3)$$

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$f(\mathrm{x})=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)..(5)$$

****

example : Find $\sqrt[3]{7}$ from $[\sqrt[3]{6}=1.82, \sqrt[3]{8}=2, \sqrt[3]{9}=2.01]$

Using Quadratic Interpolation

Solution:

1- x_0=6 $f(x_0)$=1.82 , x_1=8 $f(x_1)$=2 , x_2=9 $f(x_2)$=2.01

2- $f(6,8)=\frac{2-1.82}{8-6}=\frac{0.18}{2}=0.09$

3- $f(8,9)=\frac{2.01-2}{9-8}=\frac{0.01}{1}=0.01$

4- $f(6,8,9)=\frac{0.01-0.09}{9-6}=\frac{-0.08}{3}=-0.0267$

5- $f(7)=1.82+(7-6)*0.09+(7-6)(7-8)*-0.0267=1.9367$

****

Example (3.2)

Fit a second-order polynomial to the three points used in Example (3.1)

Solution

1-

$$x_0=1 \qquad f(x_0)=0$$
$$x_1=4 \qquad f(x_1)=1.386294$$
$$x_2=6 \qquad f(x_2)=1.791759$$

2- $f(x_0,x_1)=\frac{1.386294-0}{4-1}$=0.462098

3- $f(x_1,x_2)=\frac{1.791759-1.386294}{6-4}=\frac{0.405465}{2}$=0.2027325

4- $= \frac{0.2027325-0.462098}{6-1}$

$$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$f(x_0,x_1,x_2) = \frac{-0.2593655}{5} = -0.0518731$$

$$f(\mathrm{x}) = 0 + (x-1)*0.462098 - (x-1)(x-4)*0.0518731$$

$$f(2) = 0 + (2-1)*0.462098 - (2-1)(2-4)*0.0518731 = 0.5658442$$

****

General Form of Newton's Interpolating Polynomials

The preceding analysis can be generalized to fit an nth-order polynomial to (n + 1) data points. The nth-order polynomial is

$$y_n(x) = b_0 + b_1(x-x_0) + \cdots + b_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

As was done previously with the linear and quadratic interpolations, data points can be used to evaluate the coefficients b_0, b_1, . . . , b_n. For an nth order polynomial, n + 1 data points are required :($_0$, y_0), (x_1 , y_1), (x_n, y_n). We use these data points and the following equations to evaluate the coefficients:

$$b_0 = f(x_0)$$

$$b_1 = f(x_0,x_1)$$

$$b_2 = f(x_0,x_1,x_2)$$

.

.

.

$$b_n = f(x_0,x_1,\ldots,x_{n-1},x_n)$$

The
bracketed function evaluations are finite divided differences. For example, the first finite divided difference is represented generally as

The second finite divided difference, which represents the difference of two first divided differences, is expressed generally as

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i - x_j] - f[x_j - x_k]}{x_i - x_k}$$

Similarly, the nth finite divided difference is

$$f[x_0, x_1, \dots, x_{n-1}, x_n] = \frac{f[x_1, , \dots, x_{n-1}, x_n] - f[, x_0, \dots x_{n-2}, x_{n-1}]}{x_n - x_0}$$

****

Example (3.3)

In Example (3.2), data points at x_0 = 1, x_1 = 4, and x_2 = 6 were used to estimate ln (2) with a parabola. Now, adding a fourth point [x_3 = 5; f (x_3) = 1.609438], estimate ln (2) with a third-order Newton's interpolating polynomial.

Solution

The third-order polynomial with n = 3, is

$$y_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$

The second divided differences are

$$f[x_0, x_1, x_2] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_1, x_2, x_3] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

$$f[x_0, x_1, x_2, x_3] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

The result for $[x_0, x_1]$, $f[x_0, x_1, x_2]$, $f[x_0, x_1, x_2, x_3]$ represent the coefficients b_1, b_2, b_3 respectively. b_0 = 0, so a third-order Newton's interpolating polynomial

$$y_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$y_2(2) = 0 + 0.4620981(2 - 1) - 0.05187311(2 - 1)(2 - 4)$$
$$+0.007865529(2 - 1)(2 - 4)(2 - 6) = 0.6287686$$

This represents a relative error of 9.3%