Example (3.3) in another method

In Example (3.2), data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate $ln\ (2)$ with a parabola. Now, adding a fourth point [$x_3 = 5$; $f\ (x_3) = 1.609438$], estimate $ln\ (2)$ with a third-order Newton's interpolating polynomial.

Solution

The third-order polynomial with n = 3, is

i	x_i	$F(x_i)$	First	Second	Third
0	1	0	$f(x_1,x_0)$	$f(x_2,x_1,x_0)$	$f(x_3,x_2,x_1,x_0)$
1	4	1.386294	$f(x_2,x_1)$	$f(x_3,x_2,x_1)$	
2	6	1.791759	$f(x_3,x_2)$		
3	5	1.609438			

$$\textbf{First} = \quad f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$

$$\textbf{Second} = \quad f[x_2, x_1, x_0] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_3, x_2, x_1] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

$$f[x_3, x_2, x_1, x_0] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

$$\textbf{third} = f_3(x) = \mathrm{f}(x_0) + f[x_1, x_0](x - x_0) + \mathrm{f}(x_2, x_1, x_0)(x - x_0)(x - x_1) + \mathrm{f}(x_3, x_2, x_1, x_0)(x - x_0)(x - x_1)(x - x_2)$$

$$f_3(2) = 0 + 0.4620981(2 - 1) - 0.05187311(2 - 1)(2 - 4) + 0.007865529(2 - 1)(2 - 4)(2 - 6) = 0.6287686$$

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Example: **using the** Newton's Divided-Difference on the function f(x) = 2^x , $\mathbf{x_0}$=**-2**, $\mathbf{x_1}$=**-1**, $\mathbf{x_2}$=**0**, $\mathbf{x_3}$=1, $\mathbf{x_4}$=2, n=4

Solution:

i	x_i	$F(x_i)=2^x$	First	Second	Third	Forth
0	-2	$f(x_0)=0.25$	$f(x_1,x_0)$	$f(x_2,x_1,x_0)$	$f(x_3,x_2,x_1,x_0)$	$f(x_4,x_3,x_2,x_1,x_0)$
1	-1	$f(x_1)=0.5$	$f(x_2,x_1)$	$f(x_3,x_2,x_1)$	$f(x_4,x_3,x_2,x_1)$	
2	0	$f(x_2)=1$	$f(x_3,x_2)$	$f(x_4,x_3,x_2)$		
3	1	$f(x_3)=2$	$f(x_4,x_3)$			
4	2	$f(x_4)=4$				

$$\boldsymbol{First} = f[x_1,x_0] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0.5-0.25}{-1-(-2)} = 0.25$$

$$f[x_2,x_1] = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{1-0.5}{0-(-1)} = 0.5$$

$$f[x_3,x_2] = \frac{f(x_3)-f(x_2)}{x_3-x_2} = \frac{2-1}{1-0} = 1$$

$$f[x_4,x_3] = \frac{f(x_4)-f(x_3)}{x_4-x_3} = \frac{4-2}{2-1} = 2$$

$$\textbf{Second} = \ f[x_2,x_1,x_0] = \frac{0.5-0.25}{0-(-2)} = 0.125$$

$$f[x_3,x_2,x_1] = \frac{1-0.5}{1-(-1)} = 0.25$$

$$f[x_4,x_3,x_2] = \frac{2-1}{2-0} = 0.5$$

Third=$f[x_3,x_2,x_1,x_0] = \frac{0.25-0.125}{1-(-2)} = 0.04167$

$$f[x_4,x_3,x_2,x_1] = \frac{0.5-0.25}{2-(-1)} = 0.0833$$

$$\textbf{Forth} = f[x_4,x_3,x_2,x_1,x_0] = \frac{0.0833-0.04167}{2-(-2)} = 0.010415$$

$$f_4(x) = f(x_0) + f[x_1, x_0](x - x_0) + \mathrm{f}(x_2, x_1, x_0)(x - x_0)(x - x_1) + \mathrm{f}(x_3, x_2, x_1, x_0)(x - x_0)(x - x_1)(x - x_2) + \mathrm{f}(x_4, x_3, x_2, x_1, x_0)(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Lagrange's interpolation

Consider the original problem of interpolating (x_0, y_0),...,(x_n, y_n)

The unique interpolating polynomial of degree ≤n is given by

$$p(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x)$$

This is called the **Lagrangian interpolating polynomial**.

Example:

n= order=2

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

n= order=3

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

example :Let us apply Lagrange interpolation to the following table:

i	x_i	$f(x_i)$
0	1	12
1	3	10
2	-2	15

Solution:

We have already computed the polynomials L_0,L_1 and L_2.. So the unique degree 3 interpolating polynomial is

$$p(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x)$$
$$= 12(6 + x - x^2)/6 + 10(x^2 + x - 2)/10 - 15(x^2 - 4x + 3)/15$$
$$= -2x^2 + 7x + 7$$

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Example(2-1): Find a polynomial to interpolate by using Lagrange interpolation method $p_4(x)$

X	$f(x)$
0	1
1	3
2	2
3	5
4	4

Solution:

$$p_4(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} = \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{(0 - 1)(0 - 2)(0 - 3)(0 - 4)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} = \frac{(x - 0)(x - 2)(x - 3)(x - 4)}{(1 - 0)(1 - 2)(1 - 3)(1 - 4)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} = \frac{(x - 0)(x - 1)(x - 3)(x - 4)}{(2 - 0)(2 - 1)(2 - 3)(2 - 4)}$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} = \frac{(x - 0)(x - 1)(x - 2)(x - 4)}{(3 - 0)(3 - 1)(3 - 2)(3 - 4)}$$

$$L_4(x) = \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} = \frac{(x - 0)(x - 1)(x - 2)(x - 3)}{(4 - 0)(4 - 1)(4 - 2)(4 - 3)}$$

$$f_4(x) = 1 * L_0(x) + 3 * L_1(x) + 2 * L_2(x) + 5 * L_3(x) + 4 * L_4(x)$$

$$f_4(x) = 1 * \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{24} + 3 * \frac{(x - 0)(x - 2)(x - 3)(x - 4)}{-8} + 2 * \frac{(x - 0)(x - 1)(x - 3)(x - 4)}{8} + 5 * \frac{(x - 0)(x - 1)(x - 2)(x - 4)}{-6} + 4 * \frac{(x - 0)(x - 1)(x - 2)(x - 3)}{24}$$

****

Homework: Find a polynomial to interpolate in example (2-1) Both Newton's interpolation method and Lagrange 1 interpolation method must give the same answer

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Homework: Find a polynomial to interpolate the data. Find $f_3(2.8)$

x	f(x)
2.5	14
3.2	15
2	8
4	8

Homework: using the Newton's Divided-Difference , Find $f_3(7)$

x_i	$f(x_i)$
5	**12**
6	**13**
9	**14**
11	**16**