Fuzzy Set Theory

What is Fuzzy Set ?

- The word "fuzzy" means "vagueness". Fuzziness occurs when the boundary of a piece of information is not clear-cut.

- Fuzzy sets have been introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set.

- Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition - an element either belongs or does not belong to the set.

  Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [0, 1].

- **Example:**

  Words like young, tall, good, or high are fuzzy.

  - There is no single quantitative value which defines the term young.
  - For some people, age 25 is young, and for others, age 35 is young.
  - The concept young has no clean boundary.
  - Age 1 is definitely young and age 100 is definitely not young;
  - Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

1. Introduction

In real world, there exists much fuzzy knowledge;
Knowledge that is vague, imprecise, uncertain, ambiguous, inexact, or probabilistic in nature.

Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Humans, can give satisfactory answers, which are probably true.

However, our systems are unable to answer many questions. The reason is, most systems are designed based upon classical set theory and two-valued logic which is unable to cope with unreliable and incomplete information and give expert opinions.

We want, our systems should also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able provide solutions to many real world problems.

Fuzzy Set theory is an extension of classical set theory where elements have degrees of membership.

- **Classical Set Theory**

A Set is any well defined collection of objects. An object in a set is called an element or member of that set.

– Sets are defined by a simple statement describing whether a particular element having a certain property belongs to that particular set.

– Classical set theory enumerates all its elements using

$$A = \{ a_1 , a_2 , a_3 , a_4 , \ldots\ldots\ldots a_n \}$$

If the elements a_i $(i = 1, 2, 3, \ldots n)$ of a set **A** are subset of universal set **X**, then set **A** can be represented for all elements $x \in X$ by its **characteristic function**

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

– A set **A** is well described by a function called characteristic function.

This function, defined on the universal space **X**, assumes :

a value of **1** for those elements **x** that belong to set **A**, and

a value of **0** for those elements **x** that do not belong to set **A**.

The notations used to express these mathematically are

$$\left.\begin{array}{l} A : X \rightarrow [0, 1] \\ A(x) = 1 , \ x \text{ is a member of } A \\ A(x) = 0 , \ x \text{ is not a member of } A \end{array}\right\} \quad \textbf{Eq.(1)}$$

Alternatively, the set **A** can be represented for all elements $x \in X$ by its characteristic function $\mu_A(x)$ defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{Eq.(2)}$$

– Thus in classical set theory $\mu_A(x)$ has only the values **0** ('false') and **1** ('true''). Such sets are called **crisp sets.**

- **Fuzzy Set Theory**

  Fuzzy set theory is an extension of classical set theory where elements have varying degrees of membership. A logic based on the two truth values, *True* and *False*, is sometimes inadequate when describing human reasoning. Fuzzy logic uses the whole interval between **0** (false) and **1** (true) to describe human reasoning.

  - A **Fuzzy Set** is any set that allows its members to have different degree of membership, called **membership function**, in the interval **[0 , 1]**.
  - The **degree of membership** or truth is not same as probability;
    - fuzzy truth is not likelihood of some event or condition.
    - fuzzy truth represents membership in vaguely defined sets;
  - **Fuzzy logic** is derived from fuzzy set theory dealing with reasoning that is approximate rather than precisely deduced from classical predicate logic.
  - Fuzzy logic is capable of handling inherently imprecise concepts.
  - Fuzzy logic allows in linguistic form the set membership values to imprecise concepts like **"slightly", "quite" and "very".**
  - Fuzzy set theory defines Fuzzy Operators on Fuzzy Sets.

- **Crisp and Non-Crisp Set**

  – As said before, in classical set theory, the **characteristic function** $\mu_A(x)$ of Eq.(2) has only values **0** ('false') and **1** ('true'').
  Such sets are **crisp sets.**

  – For Non-crisp sets the characteristic function $\mu_A(x)$ can be defined.

  - The characteristic function $\mu_A(x)$ of Eq. (2) for the crisp set is generalized for the Non-crisp sets.
  - This generalized characteristic function $\mu_A(x)$ of Eq.(2) is called **membership function**.

  Such Non-crisp sets are called **Fuzzy Sets**.

  – Crisp set theory is not capable of representing descriptions and classifications in many cases; In fact, Crisp set does not provide adequate representation for most cases.

  – The proposition of Fuzzy Sets are motivated by the need to capture and represent real world data with uncertainty due to imprecise measurement.

  – The uncertainties are also caused by vagueness in the language.

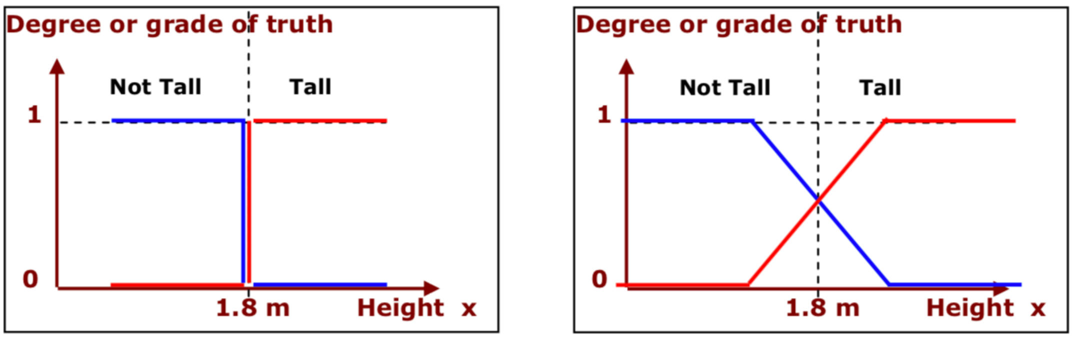
- **Representation of Crisp and Non-Crisp Set**

  Example : Classify students for a basketball team

  This example explains the grade of truth value.

  - **tall students** qualify and **not tall students** do not qualify
  - if students 1.8 m tall are to be qualified, then
    should we exclude a student who is $^{1}/_{10}$" less? or
    should we exclude a student who is 1" shorter?

  - Non-Crisp Representation to represent the notion of a tall person.



Crisp logic   Non-crisp logic

Fig. 1 Set Representation – Degree or grade of truth

A student of height 1.79m would belong to both tall and not tall sets with a particular degree of membership.

As the height increases the membership grade within the tall set would increase whilst the membership grade within the not-tall set would decrease.

- **Capturing Uncertainty**

Instead of avoiding or ignoring uncertainty, Lotfi Zadeh introduced Fuzzy Set theory that captures uncertainty.

- A fuzzy set is described by a **membership function** $\mu_A(x)$ of **A**. This membership function associates to each element $x_\sigma \in X$ a number as $\mu_A(x_\sigma)$ in the closed unit interval **[0, 1]**.

  The number $\mu_A(x_\sigma)$ represents the **degree of membership** of x_σ in **A**.
- The notation used for membership function $\mu_A(x)$ of a fuzzy set **A** is

  $$A : X \to [0, 1]$$
- Each membership function maps elements of a given universal base set **X** , which is itself a crisp set, into real numbers in **[0, 1]** .
- Example

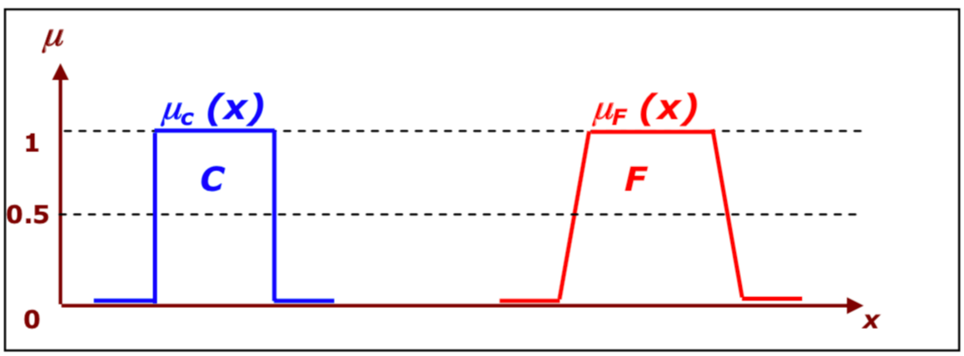


Fig. 2 Membership function of a Crisp set C and Fuzzy set F

- In the case of Crisp Sets the members of a set are :

  either out of the set, with membership of degree " **0** ",
  or in the set, with membership of degree " **1** ",

  Therefore, **Crisp Sets** $\subseteq$ **Fuzzy Sets**

  In other words, Crisp Sets are Special cases of Fuzzy Sets.

[Continued in next slide]

- **Examples of Crisp and Non-Crisp Set**

Example 1: Set of prime numbers (a crisp set)

If we consider space **X** consisting of natural numbers ≤ **12**

ie **X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}**

Then, the set of prime numbers could be described as follows.

PRIME = {x contained in X | x is a prime number} = {2, 3, 5, 6, 7, 11}

Example 2: Set of SMALL (as non-crisp set)

A Set **X** that consists of SMALL cannot be described;

for example **1** is a member of SMALL and **12** is not a member of SMALL.

Set **A**, as SMALL, has un-sharp boundaries, can be characterized by a function that assigns a real number from the closed interval from **0** to **1** to each element **x** in the set **X**.

2. Fuzzy Set

A Fuzzy Set is any set that allows its members to have different degree of membership, called membership function, in the interval **[0 , 1]**.

- **Definition of Fuzzy set**

  A **fuzzy set A**, defined in the universal space **X**, is a function defined in **X** which assumes values in the range **[0, 1]**.

  A fuzzy set **A** is written as a set of pairs **{x, A(x)}** as

  A = {{x , A(x)}} , x in the set X

  where **x** is an element of the universal space **X**, and

  A(x) is the value of the function **A** for this element.

  The value **A(x)** is the **membership grade** of the element **x** in a fuzzy set **A**.

  Example : Set **SMALL** in set **X** consisting of natural numbers $\leq$ **to 12**.

  Assume: **SMALL(1) = 1, SMALL(2) = 1, SMALL(3) = 0.9, SMALL(4) = 0.6, SMALL(5) = 0.4, SMALL(6) = 0.3, SMALL(7) = 0.2, SMALL(8) = 0.1, SMALL(u) = 0 for u >= 9.**

  Then, following the notations described in the definition above :

  Set SMALL = {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

  Note that a fuzzy set can be defined precisely by associating with each **x** , its grade of membership in **SMALL**.

- **Definition of Universal Space**

Originally the universal space for fuzzy sets in fuzzy logic was defined only on the integers. Now, the universal space for fuzzy sets and fuzzy relations is defined with three numbers.

The first two numbers specify the start and end of the universal space, and the third argument specifies the increment between elements. This gives the user more flexibility in choosing the universal space.

Example : The fuzzy set of numbers, defined in the universal space

X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

2.1 Fuzzy Membership

A fuzzy set **A** defined in the universal space **X** is a function defined in **X** which assumes values in the range **[0, 1]**.

A fuzzy set **A** is written as a set of pairs **{x, A(x)}**.

A = {{x , A(x)}} , x in the set X

where **x** is an element of the universal space **X**, and

A(x) is the value of the function **A** for this element.

The value **A(x)** is the **degree of membership** of the element **x** in a fuzzy set **A**.

The Graphic Interpretation of fuzzy membership for the fuzzy sets : Small, Prime Numbers, Universal-space, Finite and Infinite UniversalSpace, and Empty are illustrated in the next few slides.

- **Graphic Interpretation of Fuzzy Sets SMALL**

The fuzzy set SMALL of small numbers, defined in the universal space

$X = \{ x_i \} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = **FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}** , **UniversalSpace → {1, 12, 1}]**

FuzzyPlot [SMALL, AxesLable → {"X", "SMALL"}]

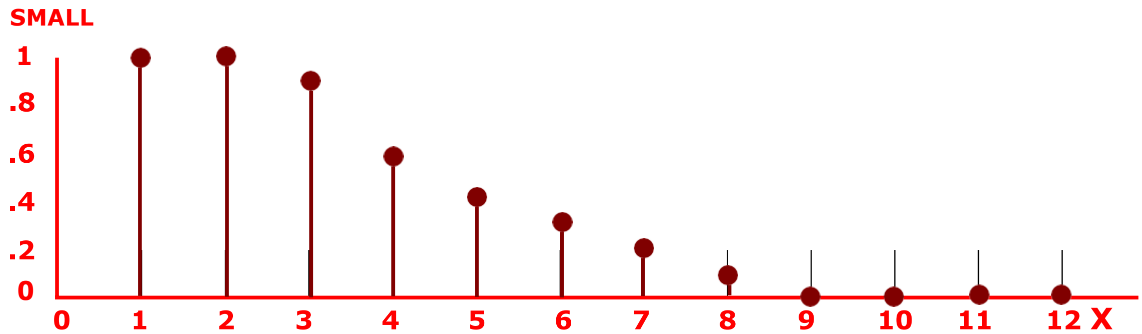


Fig Graphic Interpretation of Fuzzy Sets SMALL

- **Graphic Interpretation of Fuzzy Sets PRIME Numbers**

  The fuzzy set PRIME numbers, defined in the universal space

  X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

  SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

  The Set **PRIME** in set **X** is :

  PRIME = FuzzySet {{1, 0}, {2, 1}, {3, 1}, {4, 0}, {5, 1}, {6, 0}, {7, 1}, {8, 0}, {9, 0}, {10, 0}, {11, 1}, {12, 0}}

  Therefore **SetPrime** is represented as

  SetPrime = FuzzySet [{{1,0},{2,1}, {3,1}, {4,0}, {5,1},{6,0}, {7,1}, {8, 0}, {9, 0}, {10, 0}, {11, 1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

  FuzzyPlot [PRIME, AxesLable → {"X", "PRIME"}]

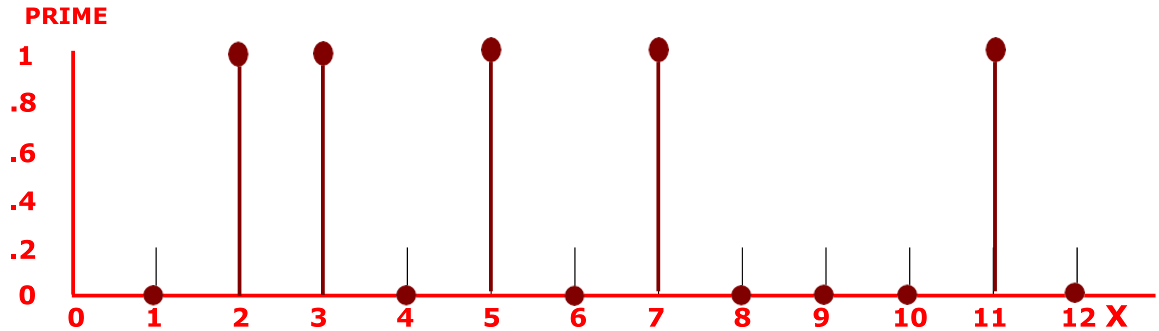


Fig Graphic Interpretation of Fuzzy Sets PRIME

- **Graphic Interpretation of Fuzzy Sets UNIVERSALSPACE**

In any application of sets or fuzzy sets theory, all sets are subsets of a fixed set called universal space or universe of discourse denoted by **X**. Universal space **X** as a fuzzy set is a function equal to **1** for all elements.

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **UNIVERSALSPACE** in set **X** is :

UNIVERSALSPACE = FuzzySet {{1, 1}, {2, 1}, {3, 1}, {4, 1}, {5, 1}, {6, 1}, {7, 1}, {8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}}

Therefore **SetUniversal** is represented as

SetUniversal = **FuzzySet [{{1,1},{2,1}, {3,1}, {4,1}, {5,1},{6,1}, {7,1}, {8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}}** , **UniversalSpace → {1, 12, 1}]**

FuzzyPlot [UNIVERSALSPACE, AxesLable → {"X", " UNIVERSAL SPACE "}]

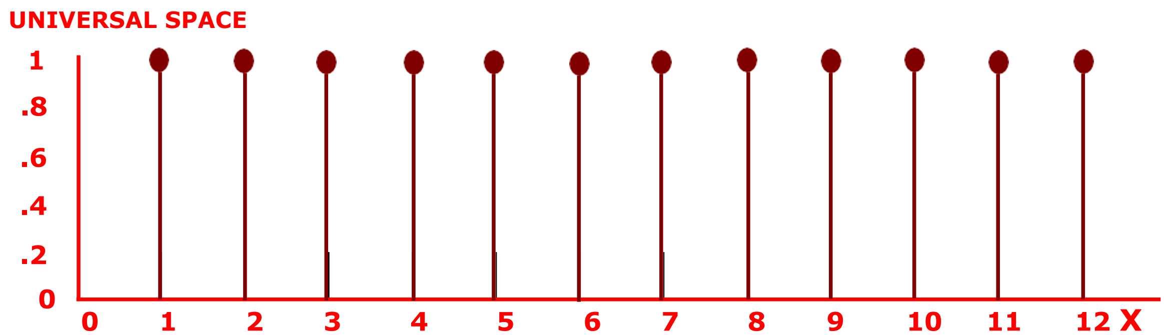


Fig Graphic Interpretation of Fuzzy Set UNIVERSALSPACE

- **Finite and Infinite Universal Space**

Universal sets can be finite or infinite.

Any universal set is finite if it consists of a specific number of different elements, that is, if in counting the different elements of the set, the counting can come to an end, else the set is infinite.

Examples:

1. Let **N** be the universal space of the days of the week.

   N = {Mo, Tu, We, Th, Fr, Sa, Su}. **N** is finite.
2. Let **M = {1, 3, 5, 7, 9, ...}.** **M** is infinite.
3. Let **L = {u | u is a lake in a city }.** **L** is finite.

   (Although it may be difficult to count the number of lakes in a city, but **L** is still a finite universal set.)

- **Graphic Interpretation of Fuzzy Sets EMPTY**

An empty set is a set that contains only elements with a grade of membership equal to **0**.

Example: Let EMPTY be a set of people, in Minnesota, older than 120.

The Empty set is also called the Null set.

The fuzzy set **EMPTY** , defined in the universal space

X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **EMPTY** in set **X** is :

EMPTY = FuzzySet {{1, 0}, {2, 0}, {3, 0}, {4, 0}, {5, 0}, {6, 0}, {7, 0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}

Therefore **SetEmpty** is represented as

SetEmpty = **FuzzySet [{{1,0},{2,0}, {3,0}, {4,0}, {5,0},{6,0}, {7,0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}** , **UniversalSpace → {1, 12, 1}]**

FuzzyPlot [EMPTY, AxesLable → {"X", " UNIVERSAL SPACE "}]

Fig Graphic Interpretation of Fuzzy Set EMPTY

2.2 Fuzzy Operations

A fuzzy set operations are the operations on fuzzy sets. The fuzzy set operations are generalization of crisp set operations. Zadeh [1965] formulated the fuzzy set theory in the terms of standard operations: Complement, Union, Intersection, and Difference.

In this section, the graphical interpretation of the following standard fuzzy set terms and the Fuzzy Logic operations are illustrated:

Inclusion : **FuzzyInclude [VERYSMALL, SMALL]**

Equality : **FuzzyEQUALITY [SMALL, STILLSMALL]**

Complement : **FuzzyNOTSMALL = FuzzyCompliment [Small]**

Union : **FuzzyUNION = [SMALL ∪ MEDIUM]**

Intersection : **FUZZYINTERSECTON = [SMALL ∩ MEDIUM]**

- **Inclusion**

Let **A** and **B** be fuzzy sets defined in the same universal space **X**.

The fuzzy set **A** is included in the fuzzy set **B** if and only if for every **x** in the set **X** we have **A(x)** ≤ **B(x)**

Example :

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The fuzzy set B SMALL

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace → {1, 12, 1}]

The fuzzy set A VERYSMALL

The Set **VERYSMALL** in set **X** is :

VERYSMALL = FuzzySet {{1, 1 }, {2, 0.8 }, {3, 0.7}, {4, 0.4}, {5, 0.2}, {6, 0.1}, {7, 0 }, {8, 0 }, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetVerySmall** is represented as

SetVerySmall = FuzzySet [{{1,1},{2,0.8}, {3,0.7}, {4,0.4}, {5,0.2},{6,0.1}, {7,0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace → {1, 12, 1}]

The Fuzzy Operation : Inclusion

Include [VERYSMALL, SMALL]

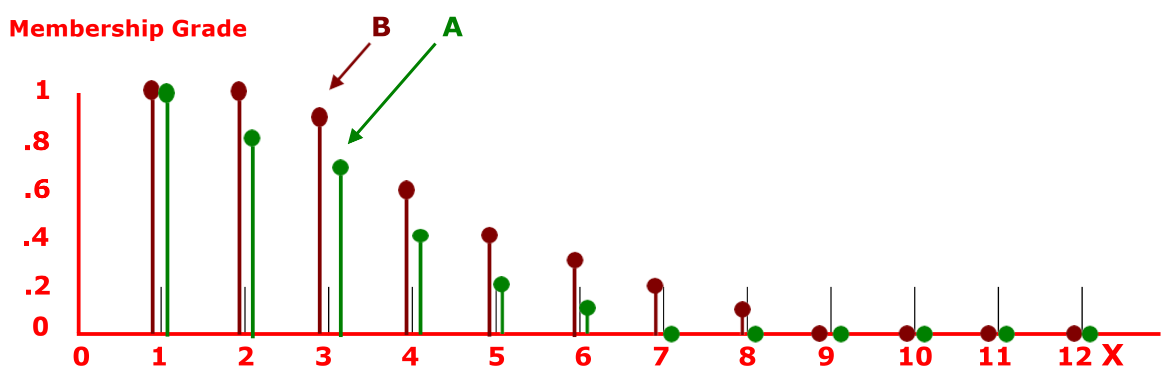


Fig Graphic Interpretation of Fuzzy Inclusion
FuzzyPlot [SMALL, VERYSMALL]

- **Comparability**

  Two fuzzy sets **A** and **B** are comparable

  if the condition **A** $\subset$ **B or B** $\subset$ **A** holds, ie,

  if one of the fuzzy sets is a subset of the other set, they are comparable.

  Two fuzzy sets **A** and **B** are incomparable

  If the condition **A** $\not\subset$ **B or B** $\not\subset$ **A** holds.

  Example 1:

  Let **A = {{a, 1}, {b, 1},{c, 0}} and**

  B = {{a, 1}, {b, 1}, {c, 1}}.

  Then **A** is comparable to **B**, since **A** is a subset of **B**.

  Example 2 :

  Let **C = {{a, 1}, {b, 1}, {c, 0.5}} and**

  D = {{a, 1}, {b, 0.9}, {c, 0.6}}.

  Then **C** and **D** are not comparable since

  C is not a subset of **D** and

  D is not a subset of **C**.

  Property Related to Inclusion :

  for all **x** in the set **X**, if **A(x)** $\subset$ **B(x)** $\subset$ **C(x), then accordingly A** $\subset$ **C.**

- **Equality**

Let **A** and **B** be fuzzy sets defined in the same space **X**.

Then **A** and **B** are equal, which is denoted **X = Y**

if and only if for all **x** in the set **X**, **A(x) = B(x).**

Example.

The fuzzy set B SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A STILLSMALL

STILLSMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The Fuzzy Operation : Equality

Equality [SMALL, STILLSMALL]

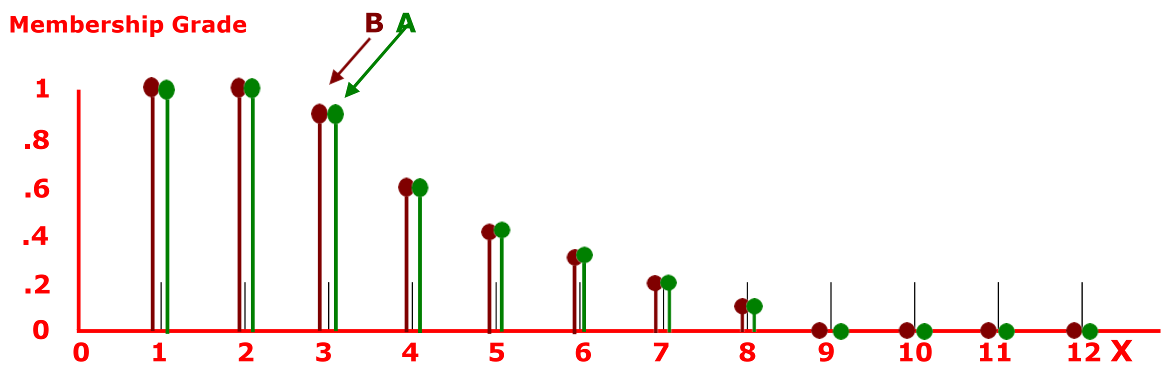


Fig Graphic Interpretation of Fuzzy Equality
FuzzyPlot [SMALL, STILLSMALL]

Note : If equality **A(x) = B(x)** is not satisfied even for one element **x** in the set **X**, then we say that **A** is not equal to **B**.

- **Complement**

Let **A** be a fuzzy set defined in the space **X**.

Then the fuzzy set **B** is a complement of the fuzzy set **A**, if and only if,

for all **x** in the set **X**, **B(x) = 1 - A(x).**

The complement of the fuzzy set **A** is often denoted by **A'** or **Ac** or $\bar{A}$

Fuzzy Complement : **Ac(x) = 1 – A(x)**

Example 1.

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set Ac NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7}, {7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The Fuzzy Operation : Compliment

NOTSMALL = Compliment [SMALL]

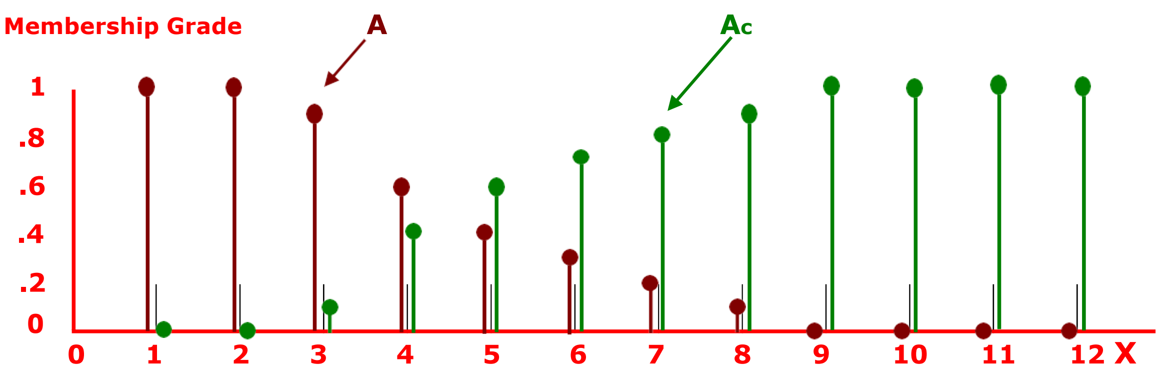


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [SMALL, NOTSMALL]

Example 2.

The empty set Φ and the universal set **X**, as fuzzy sets, are complements of one another.

$\Phi' = X$, $X' = \Phi$

The fuzzy set B EMPTY

Empty = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0}, {5, 0}, {6, 0}, {7, 0}, {8, 0}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A UNIVERSAL

Universal = FuzzySet {{1, 1 }, {2, 1 }, {3, 1}, {4, 1}, {5, 1}, {6, 1}, {7, 1}, {8, 1}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Compliment

EMPTY = Compliment [UNIVERSALSPACE]

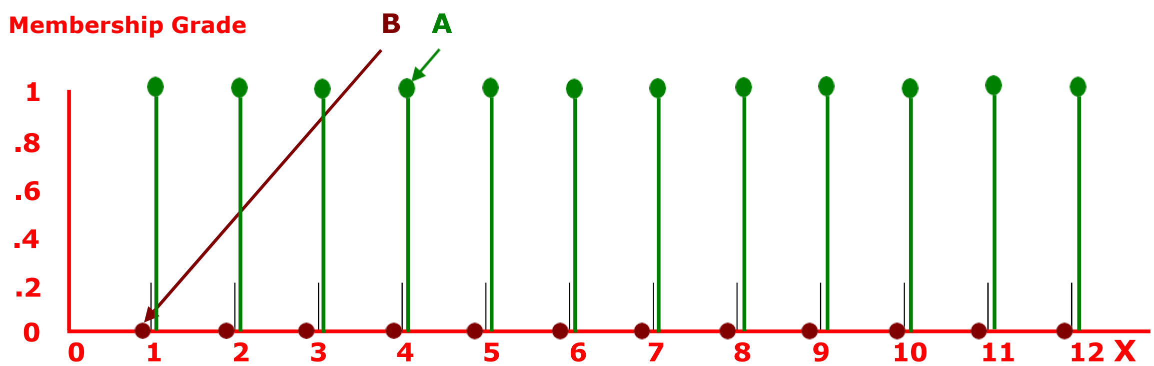


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [EMPTY, UNIVERSALSPACE]

- **Union**

Let **A** and **B** be fuzzy sets defined in the space **X**.

The union is defined as the smallest fuzzy set that contains both **A** and **B**.

The union of **A** and **B** is denoted by **A ∪ B.**

The following relation must be satisfied for the union operation :

for all x in the set X, (A ∪ B)(x) = Max (A(x), B(x)).

Fuzzy Union : (A ∪ B)(x) = max [A(x), B(x)] for all x ∈ X

Example 1 : Union of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 ∴ (A ∪ B)(x) = max [0.6, 0.4] = 0.6

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12,0}}

The fuzzy operation : Union

FUZZYUNION = [SMALL ∪ MEDIUM]

SetSmallUNIONMedium = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

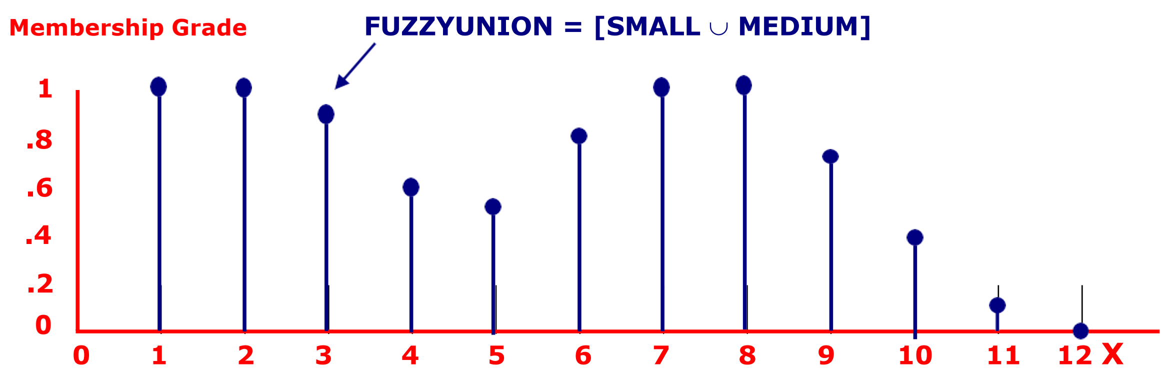


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [UNION]

The notion of the union is closely related to that of the connective "or".

Let **A** is a class of "Young" men, **B** is a class of "Bald" men.

If "David is Young" or "David is Bald," then David is associated with the union of **A** and **B.** Implies David is a member of **A ∪ B.**

- **Intersection**

Let **A** and **B** be fuzzy sets defined in the space **X**. Intersection is defined as the greatest fuzzy set that include both **A** and **B**. Intersection of **A** and **B** is denoted by **A ∩ B.** The following relation must be satisfied for the intersection operation :

for all x in the set X, (A ∩ B)(x) = Min (A(x), B(x)).

Fuzzy Intersection : (A ∩ B)(x) = min [A(x), B(x)] for all x ∈ X

Example 1 : Intersection of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 ∴ (A ∩ B)(x) = min [0.6, 0.4] = 0.4

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy operation : Intersection

FUZZYINTERSECTION = min [SMALL ∩ MEDIUM]

SetSmallINTERSECTIONMedium = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.4}, {6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace → {1, 12, 1}]

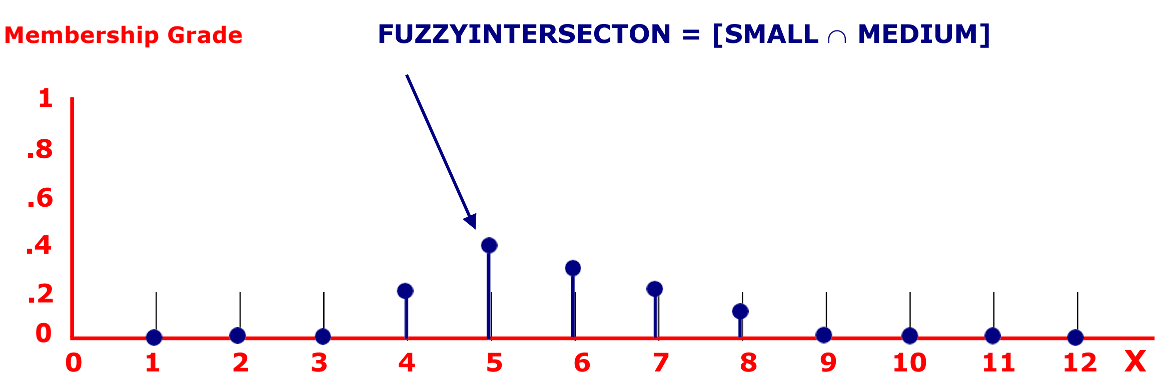


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [INTERSECTION]

- **Difference**

Let **A** and **B** be fuzzy sets defined in the space **X**.

The difference of **A** and **B** is denoted by **A ∩ B'**.

Fuzzy Difference : **(A - B)(x) = min [A(x), 1- B(x)] for all x ∈ X**

Example : Difference of **MEDIUM and SMALL**

The fuzzy set A MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy set B SMALL

MEDIUM = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0.7 }, {10, 0.4 }, {11, 0}, {12, 0}}

Fuzzy Complement : Bc(x) = 1 – B(x)

The fuzzy set Bc NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7}, {7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Difference by the definition of **Difference**

FUZZYDIFFERENCE = [MEDIUM ∩ SMALL']

SetMediumDIFFERECESmall = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.5}, {6,0.7}, {7,0.8}, {8, 0.9}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

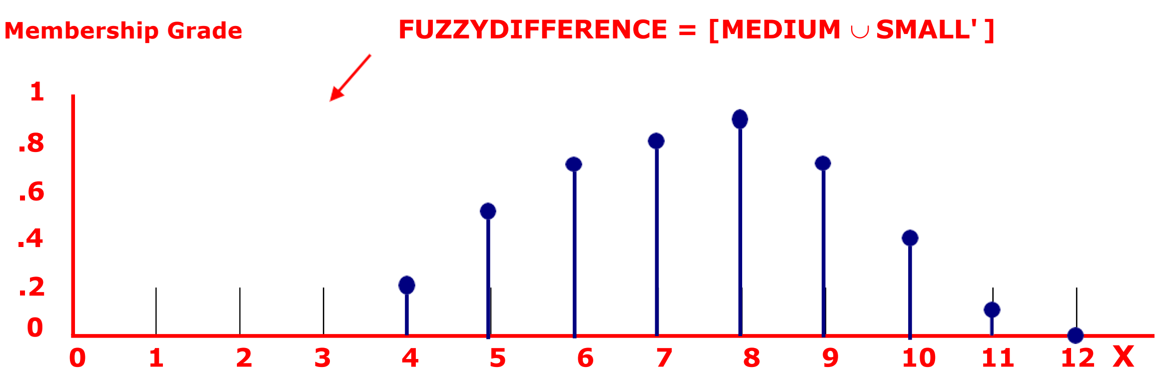


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [UNION]

Membership Functions

The membership function $\mu_A(x)$ describes the membership of the elements x of the base set X in the fuzzy set A, whereby for $\mu_A(x)$ a large class of functions can be taken. Reasonable functions are often piecewise linear functions, such as triangular or trapezoidal functions.

The grade of membership $\mu_A(x_0)$ of a membership function $\mu_A(x)$ describes for the special element $x=x_0$, to which grade it belongs to the fuzzy set A. This value is in the unit interval [0,1]. Of course, x_0 can simultaneously belong to another fuzzy set B, such that $\mu_B(x_0)$ characterizes the grade of membership of x_0 to B. This case is shown in figure-12

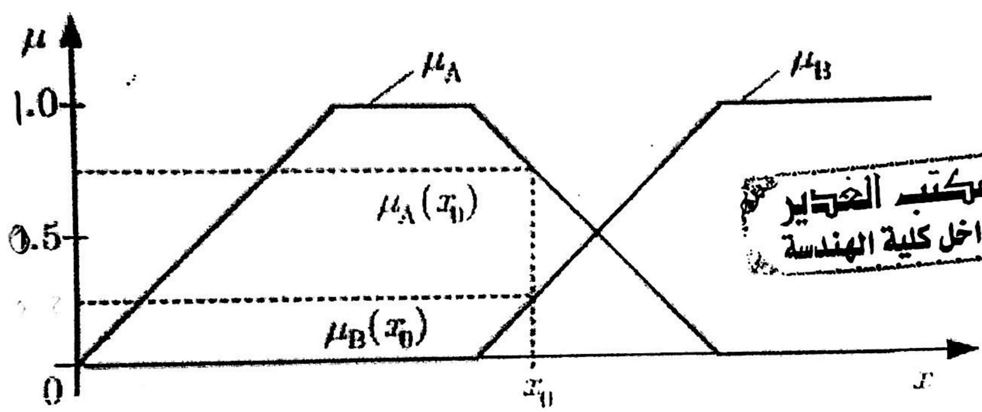


Figure-12 Membership Grades of x_0 in the Sets A and B:
$\mu_A(x_0)$ =0.75 and $\mu B(xo)$ =0.25

Fuzzy subset can also be discrete, the next example illustrates the fuzzy operations on discrete (sub)set.

Example 12: Let *A* and ***B*** be discrete fuzzy subset of X = {-3,-2,-1,0,1,2,3}.
If A={(-3, 0.0), (-2, 0.3), (-1, 0.6), (0, 1.0), (1, 0.6), (2, 0.3), (3, 0.0)}, and B={(-3, 1.0), (-2, 0.5), (-1, 0.2), (0, 0.0), (1, 0.2), (2, 0.5), (3, 1.0)}, then

$$\mu_{A\vee B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$\mu_{A\vee B}(x)$ ={(-3, 1.0), (-2, 0.5), (-1, 0.6), (0, 1.0), (1, 0.6), (2, 0.5), (3,1.0)} and

$$\mu_{A \wedge B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$\mu_{A \wedge B}(x)$ ={(-3, 0.0), (-2, 0.3), (-1, 0.2), (0, 0.0), (1, 0.2), (2, 0.3), (3,0.0)}

And the negations of A and B are

¬A={(-3, 1.0), (-2, 0.7), (-1, 0.4), (0, 0.0), (1, 0.4), (2, 0.7), (3, 1.0)},

and

¬B={(-3, 0.0), (-2, 0.5), (-1, 0.8), (0, 1.0), (1, 0.8), (2, 0.5), (3, 0.0)}

A graphical representation is shown in figure-13

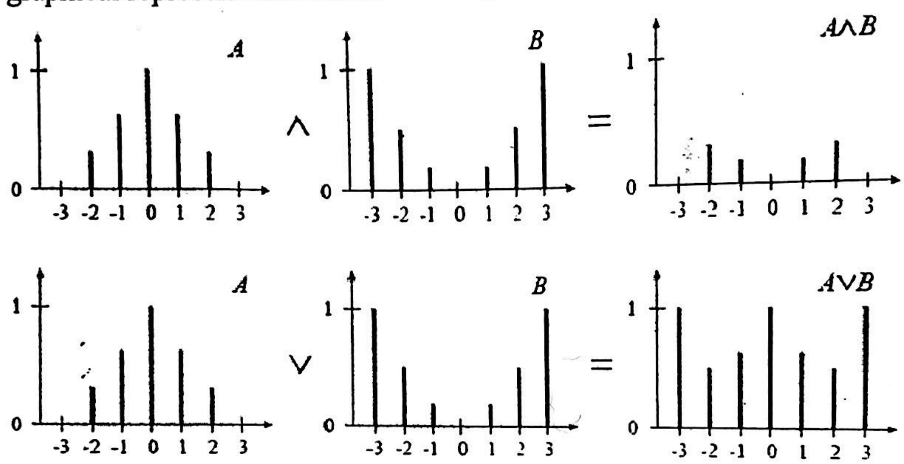


Figure-13 A graphical representation of example 12

Example 13: Let us assume that

A = "x considerable larger than 10" , B = "x approximately 11,"

characterized by

$$A = \{(x, \mu_A(x)) \mid x \in X\}$$

where

$$\mu_A(x) = \begin{cases} 0 & x \le 10 \\ 1 + (x-10)^{-2})^{-1} & x > 10 \end{cases}$$

$$B = \{(x, \mu_B(x)) \mid x \in X\}$$

where

$$\mu_B(x) = (1 + (x - 11)^4)^{-1}$$

Then

$$\mu_{A \cap B}(x) = \begin{cases} \min[(1 + (x - 10)^{-2})^{-1}, (1 + (x - 11)^4)^{-1} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

(x considerably larger than 10 and approximately 11)

$$\mu_{A \cup B}(x) = \max[(1 + (x - 1)^{-2})^{-1}, (1 + (x - 11)^4)^{-1}, \quad x \in X$$

Figure-14 depicts the above.

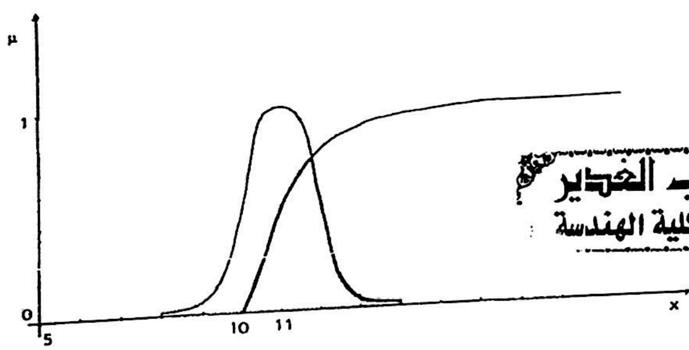


Figure-14 Union and intersection of fuzzy sets

Types of Membership Functions

In principle any function of the form A:X→[0,1] describes a membership function associated with a fuzzy set A that depends not only on the concept to be represented, but also on the context in which it is used. The graphs of the functions may have different shapes and may have specific properties. Whether a particular shape is suitable can be determined only in the application context. In certain cases, however, the meaning semantics captured by fuzzy sets is not too sensitive to variations in the shape, and simple functions are convenient.

In many practical instances fuzzy sets can be represented explicitly by families of parameterized functions, the most common being the following:

1. Γ and L open shoulder functions
2. Triangular function
3. Trapezoidal Function
4. Gaussian Function
5. S-Function

The membership function definitions for the above mentioned common membership functions are given in the following sections:

1- Γ and L open shoulder functions

Initially we will define two so called open membership functions. These are characterized as being non-decreasing and having values inside 0 and 1 only within a bounded interval. Firstly, we have functions with open right shoulders, Γ: X→[0, 1], and defined by two parameters according to the following:

$$\Gamma(x;\alpha,\beta) = \begin{cases} 0, & x < \alpha \\ (x-\alpha)/(\beta-\alpha), & \alpha \le x \le \beta \\ 1, & x > \beta. \end{cases}$$

Correspondingly, we have functions with open left shoulders, L: X→[0, 1], defined by:

$$L(x;\alpha,\beta)=\begin{cases}1, & x<\alpha\\ (\beta-x)/(\beta-\alpha), & \alpha\le x\le\beta\\ 0, & x>\beta.\end{cases}$$

These two functions are shown in figure-15 and figure-16

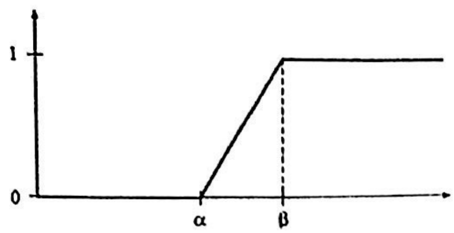


Figure-15 Γ -membership function

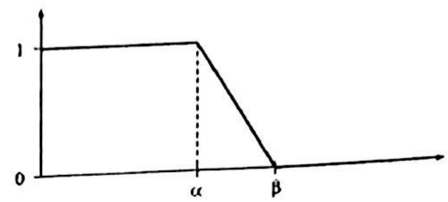


Figure-16 L -membership function

2- Triangular function

The triangular membership function Λ: X→[0, 1], is given by three parameters according to:

$$\Lambda(x;\alpha,\beta,\gamma)=\begin{cases}0, & x<\alpha\\ (x-\alpha)/(\beta-\alpha), & \alpha\le x\le\beta\\ (\gamma-x)/(\gamma-\beta), & \beta\le x\le\gamma\\ 0, & x>\gamma.\end{cases}$$

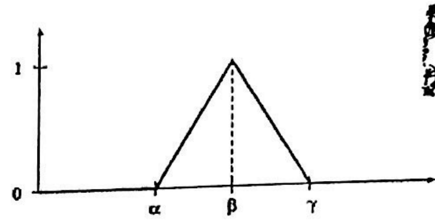


Figure-17 Λ- membership function

Example 14: In control applications, it is common to use linguistic variables like *negative big* (NB), *negative small* (NS), *zero* (ZO), *positive small* (PS) and *positive big* (PB) to express measurement values in a fuzzy way. The arrangement with the membership functions to cover the entire measurement space is then obtained using shoulders on respectively leftmost and rightmost functions and triangle function for the rest as shown in figure-18.

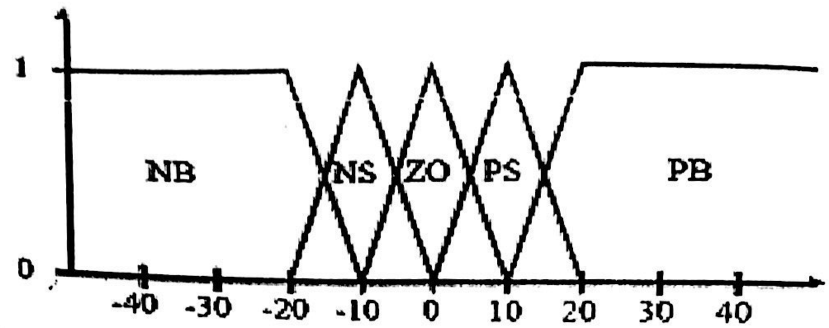


Figure-18 the arrangement of membership functions for example 14

3- Trapezoidal function

The trapezoidal membership function, Π: X → [0, 1], (shown in figure-19), is given by four parameters according to

$$\Pi(x;\alpha,\beta,\gamma,\delta) = \begin{cases} 0, & x < \alpha \\ (x-\alpha)/(\beta-\alpha), & \alpha \le x < \beta \\ 1, & \beta \le x \le \gamma \\ (\delta - x)/(\delta-\gamma), & \gamma < x \le \delta \\ 0, & x > \delta. \end{cases}$$

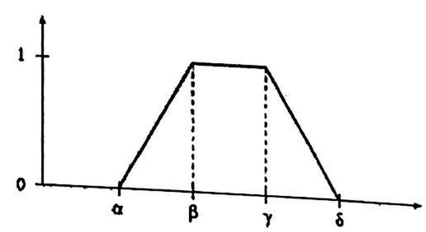


Figure-19 Π - membership function

4- Gaussian function

The membership definition for a Gaussian function G: → [0, 1] is given by two parameters as :

$$G(x:\alpha,\beta) = e^{-\beta(x-\alpha)^2}$$

Where α is the midpoint and β reflects the slop value. Note that β must be positive and that the function never reaches zero. The Gaussian function can also be extended to have different left and right slops. We then have three parameters in

$$G(x:\alpha,\beta_l,\beta_r)=\begin{cases} e^{-\beta_l(x-\alpha)^2}, & x\le\alpha \\ e^{-\beta_r(x-\alpha)^2}, & x>\alpha \end{cases}$$

Where β_l and β_r are, respectively, left and right slopes.

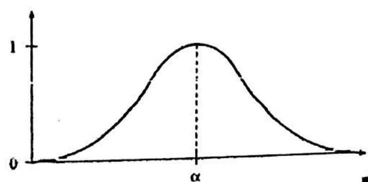


Figure-20 G-membership function

5- Sigmoidal membership function (S-function)

The typical sigmoidal membership function or S-function (shown in figure-21), σ: X→ [0, 1], needs two parameters, and can be expressed as

$$\sigma(x;\alpha,\beta)=\frac{1}{1+e^{-\beta(x-\alpha)}};$$

Where α is the midpoint and β is the slop value at the inflexion point. Similarly, as in Gaussian, β must be positive. This S-function never reaches neither 0 nor 1.

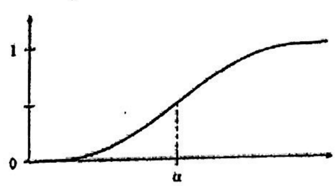


Figure-21 σ - function

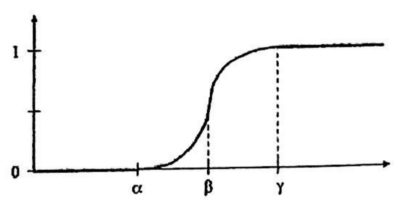


Figure-22 S-function

Zadeh defined an S-function (shown in figure-22), using polynomials rather than exponential according to:

$$S(x;\alpha,\beta,\gamma)=\begin{cases} 0, & x\le\alpha \\ 2((x-\alpha)/(\gamma-\alpha))^2, & \alpha<x\le\beta \\ 1-2((x-\gamma)/(\gamma-\alpha))^2, & \beta<x\le\gamma \\ 1, & x>\gamma. \end{cases}$$

Where β = (α + γ)/2. this function again is more efficient when considering implementation.

Fuzzy Control Systems

The most widespread use of fuzzy logic today is in *fuzzy control* applications. Across section of applications that have successfully used fuzzy control includes:

Environmental Control

- Air conditioners
- Humidifiers

Domestic Goods

- Washing machines/Dryers
- Vacuum cleaners
- Microwave ovens
- Refrigerators

Consumer Electronics

- Television
- Photocopier
- Video camera – auto focus

Automotive Systems

- Automatic Gearbox
- Four – Wheel steering
- Seat/Mirror control system

Process of Fuzzy Control

Before the development of fuzzy logic controller (FLC) systems there were essentially two alternatives to process control: ***A process was controlled by either a human operator or a computerized direct digital control system (DDC).*** The function of such a direct digital control system which is shown in figure-27 can be described as follows:

The problem consists in dimensioning a control algorithm based on the error vector $e = (e_1, e_2, \ldots, e_p)$ that generates an output vector $u = (u_1, u_2, \ldots, u_r)$ to the process so that the output vector $y = (y_1, y_2, \ldots, y_p)$ of the process is close to or eventually equal to the set point vector $r = (r_1, r_2, \ldots, r_p)$. In other words, we want to control the process by means of an algorithm of the following general form

$$u[(k + 1)T] = f(u[kT], u[(k - 1)T], \ldots, u[0], e(k+1)T], e[kT], \ldots, e[0])$$

where $k = 1,2, \ldots,$ and T is the sampling time.

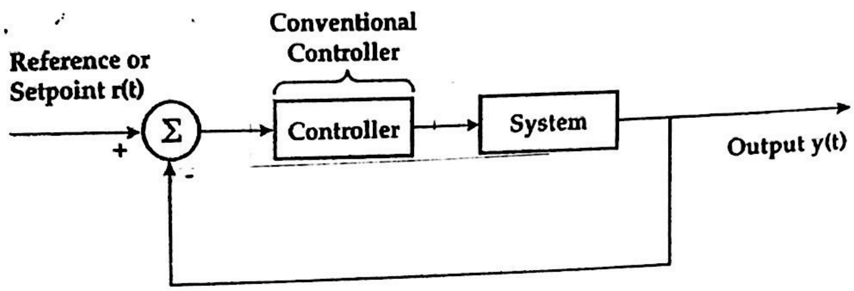


Figure -27 DDC control system

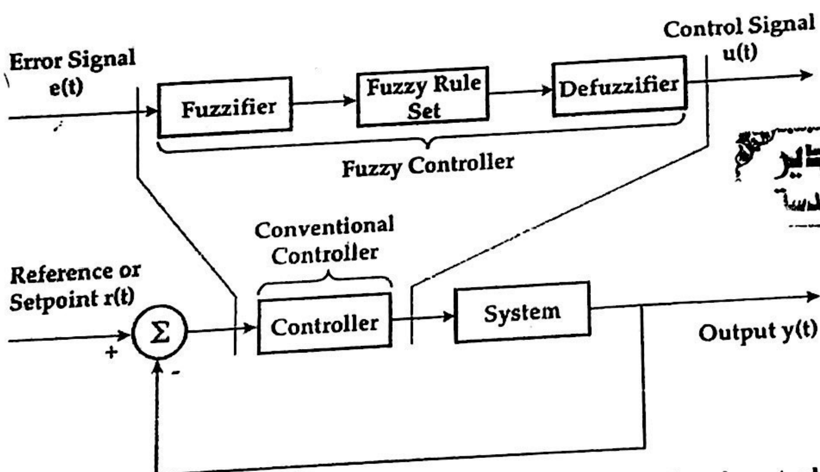


Figure -28 The Fuzzy controller and its relation to a conventional control loop

The structure of a fuzzy logical controller is depicted in figure 28. The process of fuzzy control can roughly be described as shown in figure 29.

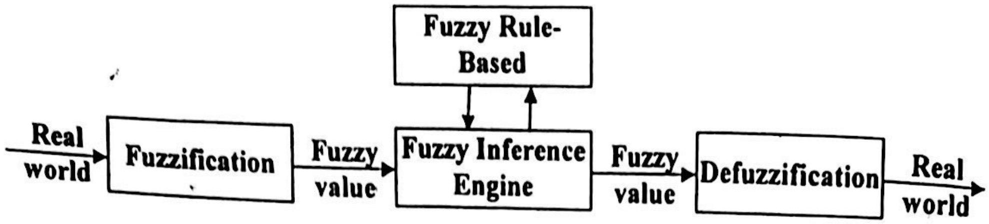


Figure -29 The Fuzzy controller System

1. Fuzzification

The fuzzification is defined as a mapping from a real-word point to a fuzzy set using a specific membership function that is described previously.

Thus, the first step is to convert the measured signal x (which might be the error signal in a control system) into a set of fuzzy variables. It is done by giving values (these will be our fuzzy variables) to each of a set of membership functions. The values for each membership function μ(x) are determined by the original signal x and the shape of the membership. For example, let us say that the fuzzifier splits the signal x into five fuzzy levels as follows (see figure 30):

x is large positive:	LP
x is medium positive:	MP
x is small:	S
x is medium negative:	MN
x is large negative:	LN

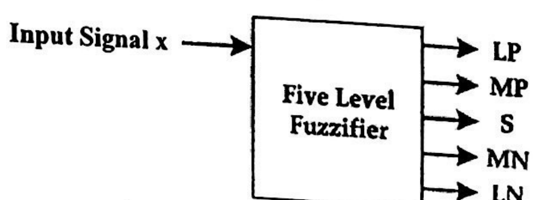


Figure-30 Five level Fuzzifier

As the input to the fuzzifier changes in the range -10v to +10v, then the corresponding fuzzy variables will also change.

A practical fuzzifier would have a measured signal sensor at its input and would provide at its output the values (fuzzy variables) corresponding to the membership functions. For example, if a sensor signal with an output voltage of 2v is applied to the five level fuzzifier, the resulting set of fuzzy variables is (as shown in figure 31):

$\mu_{LN} = 0$
$\mu_{MN} = 0$
$\mu_{S} = 0.6$
$\mu_{MP} = 0.4$
$\mu_{LP} = 0$

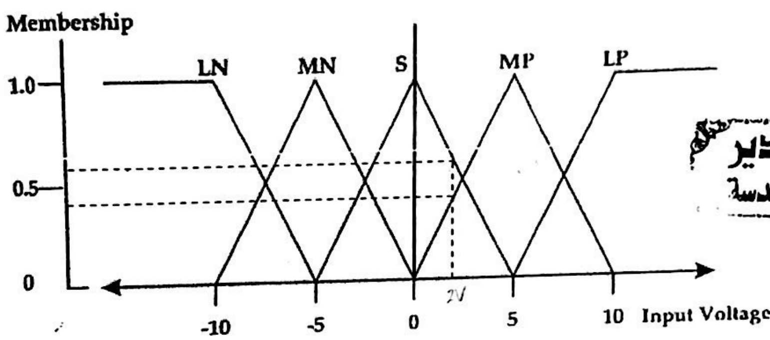


Figure-31 The complete set of membership functions for five level fuzzification

2. ***Fuzzy Rule-Base***

A fuzzy rule-base consists of a set of fuzzy IF-THEN rules. The control rules are defined as fuzzy conditional statements of this type.

As an example. the fuzzy IF-THEN rule can be used to control the speed (SP) of a motor by changing the speed (CS). This can take the following form:

"If SP is PB then CS is NB"

3. ***Fuzzy Inference Engine***

In fuzzy inference engine, fuzzy logic principles are used to combine the fuzzy IF-THEN rules in the fuzzy rule-base into a mapping from one fuzzy set to another fuzzy set. The min-max compositional rule of inference is then used to derive fuzzy control statements from observed observations of the states of the process. If several rules are combined by "else," this is interpreted as the union operator "max". for example:

IF {error S} AND {output_rate LP} THEN {control LN}

OR IF {error S} AND {output_rate LN} THEN {control LP}

4. *Defuzzification*

The defuzzification represents the last step in building a fuzzy logic process. Defuzzification can be defined as a mapping from a fuzzy values that results from the previous stages into a real-word value. In other word, It combines the fuzzy variables to give corresponding real (crisp or non-fuzzy) signal which can then be used to perform some action. For example a five level defuzzifier block which is shown in figure 32, will have inputs corresponding to the following five actions:

LP : Output signal large (positive)
MP : Output medium (positive)
S : Output signal small
MN: Output signal medium (negative)
LN : Output signal large (negative)

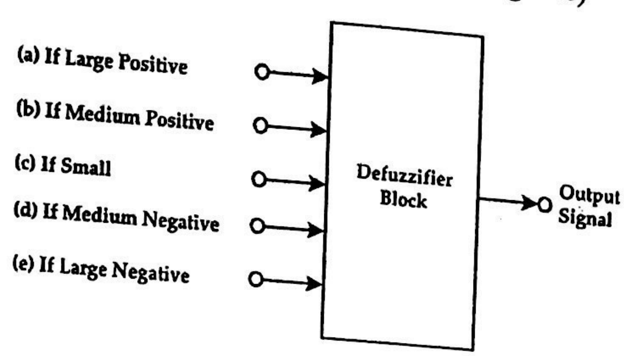


Figure -32 Ablock diagram of defuzzifier

The defuzzifier combines the information in the fuzzy inputs to obtain a single crisp (non-fuzzy) output variable. There are a number of defuzzification methods, such as, center average, maximum defuzzifier, and center of gravity. The simplest and the most used one is the center of gravity. It works as like this: If the fuzzy level LP, … , LN have membership values that are labeled $\mu_1, \ldots, \mu_5$, then the crisp output signal u is defined as:

$$COG = u = \frac{\sum_{i=1}^{5} u_i \mu_i}{\sum_{i=1}^{5} \mu_i}$$

For example, the values of the u_i of the membership functions shown in figure 31 are, $u_1 = 10V$, $u_2 = 5V$, $u_3 = 0V$, $u_4 = -5V$, and $u_5 = -10V$, and corresponding to the central points of the fuzzy classes LP; MP; S; MN; LN at the input to the defuzzifier.

Proportional plus derivative fuzzy controller

The fuzzy proportional controller can be extended to cover integral and derivative control. Here we outline just the derivative control extension. In this case the fuzzy logic operates on the error signal e(t) and the derivative of the output signal ***dy(t)/dt*** and produces an output from its defuzzifier which is the control signal ***u(t)***. the fuzzy logic controller bases its actions on the two signals, the error and the rate of change of the output. The output derivative is either available as a direct measurement from the system or by using an observer of the system states.

Design principle of fuzzy logic controller

To design a fuzzy logic controller that has multi inputs and single output, the following steps must be considered:

1- Determine the inputs and the output.

2- Put the control knowledge into rule-base, which includes.

2.1 Linguistic description, that describes the input and output linguistic variable.

2.2 Quantify the linguistic variable: linguistic variables assume "linguistic values", such as: LP, MP, S, MN, LN. next the designer determine the type of membership functions used to fully quantify these fuzzy sets so that the user may automate the control rules specified by expert.

2.3 Specify the set of rules. A convenient way to list all possible rules for the case where there are not many inputs to the fuzzy controller is to use tabular representation.

3- Matching: Determine which rule to use: this step is done using the inference mechanism, which involves two steps:

3.1 The IF parts of all the rules are compared to the controller inputs to determine which rules apply to the current situation.

3.2 Next, THEN parts (what control action to take) are determined using the rules that have been determined to apply at the current time.

However, the steps used by the user to calculate the control action are:

a- Calculate error and change in error. According to that define the fuzzy sets.

b- Calculate the degree of activation for each rule, this can be achieved by implementing the IF pats of all fuzzy rules (finding the minimum values of membership functions for error and change in error.

c- Calculate the control vector (u_i) for each rule.

d- Calculate the control action by using the defuzzification operation.

The center of gravity (CoG) defuzzification method can be used

$$u = \frac{\sum_{n=1}^{N} I_n \mu_n}{\sum_{n=1}^{N} \mu_n}$$

Where, I_n is the value of the interval, n= 1, 2, …, N, and N is the total no. of intervals.

Example 22: The following table illustrates the linguistic sets and its membership degree corresponding to each interval, suppose that the number of intervals are 21. The linguistic variables are quantified into five linguistic values (fuzzy sets). NL stands for (Negative Large), NS stands for "Negative small", Z stands for "Zero", PS stands for " Positive Small", PL stands for "Positive Large".

Find the control action if the error is (- 0.9) and the change in error is (0.1) using the fuzzy rules and the rules sets given:

Fuzzy rules are:

	NB	NS	Z	PS	PB
NB	PB	PB	PB	PS	Z
NS	PB	PS	PS	Z	NS
Z	PB	PS	Z	NS	NB
PS	PS	Z	NS	NS	NB
PB	Z	NS	NB	NB	NB

The rules set:

Interval	Measured signal	Scaled signal	PB	PS	Z	NS	NB
1	-1	-10	0	0	0	0	1
2	-.9	-9	0	0	0	0	1
3	-.8	-8	0	0	0	0	1
4	-.7	-7	0	0	0	0	1
5	-.6	-6	0	0	0	.25	.75
6	-.5	-5	0	0	0	.5	.5
7	-.4	-4	0	0	0	.75	.25
8	-.3	-3	0	0	0	1	0
9	-.2	-2	0	0	.25	.75	0
10	-.1	-1	0	0	.5	.5	0
11	0	0	0	0	.75	.25	0
12	.1	1	0	0	1	0	0
13	.2	2	0	.25	.75	0	0
14	.3	3	0	.5	.5	0	0
15	.4	4	0	.75	.25	0	0
16	.5	5	0	1	0	0	0
17	.6	6	.25	.75	0	0	0
18	.7	7	.5	.5	0	0	0
19	.8	8	.75	.25	0	0	0
20	.9	9	1	0	0	0	0
21	1	10	1	0	0	0	0

Solution:

Error (E) = - 0.9   scaled E = -9   Then it is NB

Change in error (CE) = 0.1   scaled CE = +1   Then it is PS and Z

Therefore two rules will be fired:

IF E is NB AND CE is PS THEN U is PS

IF E is NB AND CE is Z THEN U is PB

For the first rule $\mu_E = 1$ and $\mu_{CE} = 0.3$ thus, $\mu_u = 0.3$

For the second rule $\mu_E = 1$ and $\mu_{CE} = 0.7$ thus, $\mu_u = 0.7$

Now, the control action will be:

U = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, .3, .3, .3, .3, .3, .5, .7, .7, .7, .7}

Finally, we apply center of gravity (CoG) defuzzification in order to obtain final crisp output:

$$U = \frac{(0*-10)+(0*-9)+(0*-8)+...+(.3*1)+(.3*2)+...+(.5*6)+...+(.7*10)}{0+0+...+.3+...+.3+.5+.7+...+.7} = \frac{31.3}{4.8} = 6.52$$