Electrical Circuit-II

8^{th} Lecture

Resonance

(Part 1)

By:

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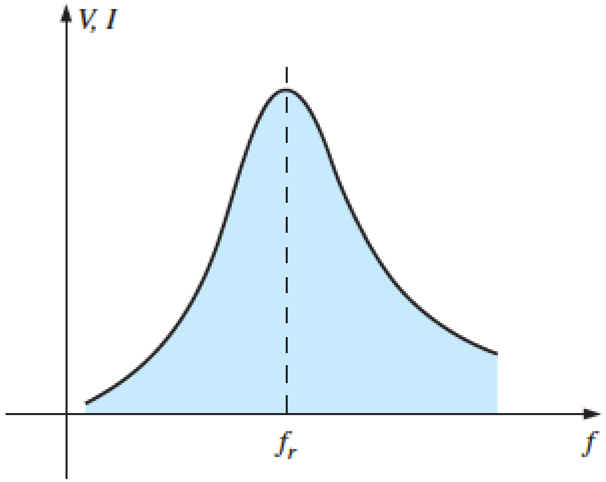
Ref: Robert L. Boylestad, *INTRODUCTORY CIRCUIT ANALYSIS*, Pearson Prentice Hall, Eleventh Edition, 2007

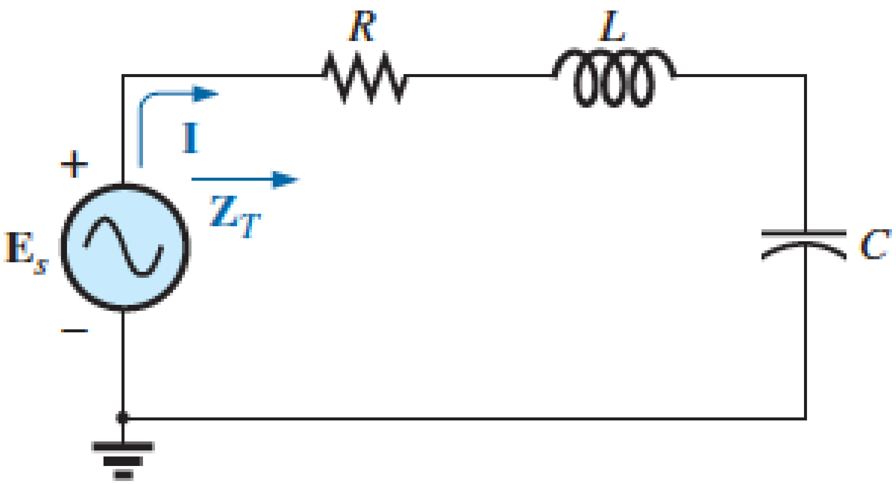
Resonance

Resonant (or tuned) circuit, which is fundamental to the operation of a wide variety of electrical and electronic systems in use today. The resonant circuit is a combination of R, L, and C elements having a frequency response characteristic similar to the one appearing bellow. Note in the figure that the response is a maximum for the frequency (fr) , decreasing to the right and left of this frequency. In other words, for a particular range of frequencies, the response will be near or equal to the maximum. The frequencies to the far left or right have very low voltage or current levels and, for all practical purposes, have little effect on the system's response. The radio or television receiver has a response curve for each broadcast station. When the receiver is set (or tuned) to a particular station, it is set on or near the frequency (fr), and said to be in a state of resonance.



Series Resonant Circuit

Consider we have the circuit shown



The total impedance of this circuit:

$$Z_T = R + jX_L - jX_C = R + j(X_L - X_C)$$

The resonant conditions described in the introduction occurs when:

$$X_L = X_C$$

The total impedance at resonance is then:

$$Z_{T_s} = R$$

The resonant frequency

The resonant frequency can be determined in terms of the inductance and capacitance by examining the defining equation for resonance:

$$X_L = X_C$$

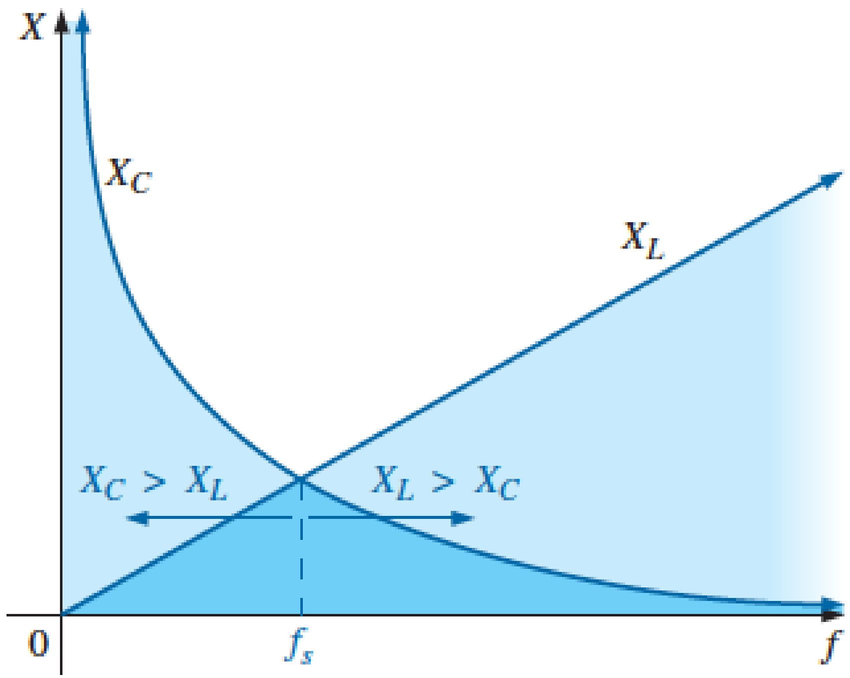
Substituting yields

$$\omega L = \frac{1}{\omega C} \quad \text{and} \quad \omega^2 = \frac{1}{LC}$$

and
$$\boxed{\omega_s = \frac{1}{\sqrt{LC}}}$$

or
$$\boxed{f_s = \frac{1}{2\pi\sqrt{LC}}}$$

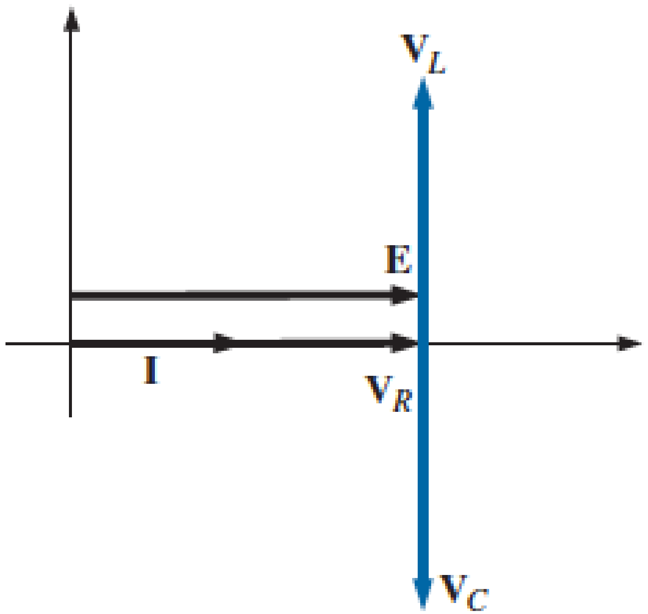
f = hertz (Hz)
L = henries (H)
C = farads (F)



The resonant frequency

The current through the circuit at resonance is

$$\mathbf{I} = \frac{E \angle 0^\circ}{R \angle 0^\circ} = \frac{E}{R} \angle 0^\circ$$



The average power to the resistor at resonance is equal to

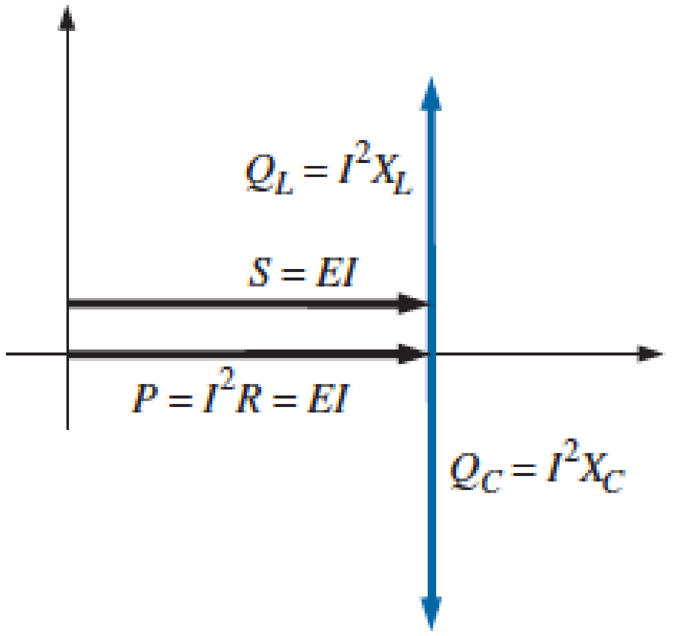
$$I^2R$$

reactive power to the capacitor and inductor are

$$I^2X_C \text{ and } I^2X_L$$

The total power is equal to the average power dissipated by the resistor since

$$Q_L = Q_C$$



The power factor of the circuit at resonance is

$$F_p = \cos\theta = \frac{P}{S}$$

$$\boxed{F_{p_s} = 1}$$

The resonant frequency

- The total reactance of the circuit is zero $(X_L - X_C = 0)$
- The circuit impedance is minimum $(Z_T = R)$
- The circuit current is maximum $(I = \frac{E}{Z_T} = \frac{E}{R})$
- The circuit power factor angle is (0^o) and the power factor is (1)
- At resonance $(w^2 LC = 1)$

The Quality Factor (Q)

The **quality factor** *Q* of a series resonant circuit is defined as the ratio of the reactive power of either the inductor or the capacitor to the average power of the resistor at resonance; that is,

$$Q_s = \frac{\text{reactive power}}{\text{average power}}$$

Substituting for an inductive reactance at resonance gives us

$$Q_s = \frac{I^2X_L}{I^2R}$$

$$Q_s = \frac{X_L}{R} = \frac{\omega_s L}{R}$$

$$Q_s = \frac{\omega_s L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi}{R}\left(\frac{1}{2\pi\sqrt{LC}}\right)L$$

$$= \frac{L}{R}\left(\frac{1}{\sqrt{LC}}\right) = \left(\frac{\sqrt{L}}{\sqrt{L}}\right)\frac{L}{R\sqrt{LC}}$$

$$\omega_s = 2\pi f_s$$

$$f_s = \frac{1}{2\pi\sqrt{LC}}$$

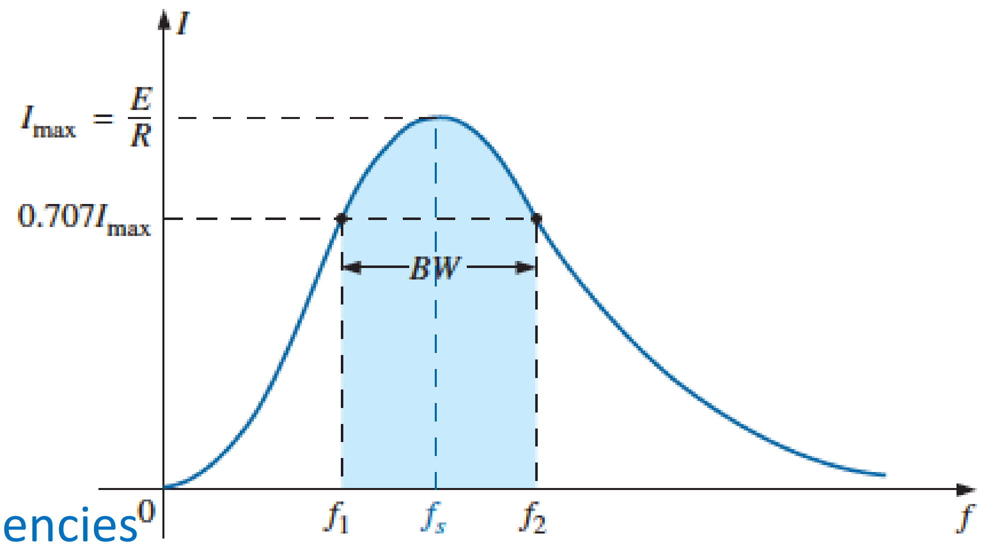
$$Q_s = \frac{1}{R}\sqrt{\frac{L}{C}}$$

For circuits where Qs ≥ 10, a widely accepted approximation is that the resonant frequency bisects the bandwidth and that the resonant curve is symmetrical about the resonant frequency.

Bandwidth (BW)

There is a definite range of frequencies at which the current is near its maximum value and the impedance is at a minimum. Those frequencies corresponding to 0.707 of the maximum current are called the band frequencies, cutoff frequencies, half-power frequencies, or corner frequencies.

They are indicated by *f1* and *f2*



The range of frequencies between the two is referred to as the bandwidth (BW) of the resonant circuit.

Half-power frequencies are those frequencies at which the power delivered is one-half that delivered at the resonant frequency; that is,

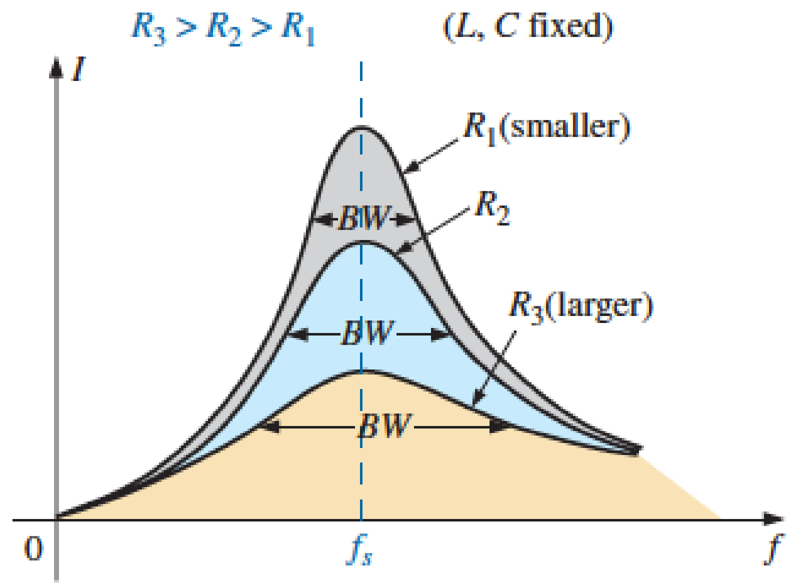
$$P_{HPF} = \frac{1}{2}P_{max}$$

$$P_{max} = I_{max}^2 R$$

$$P_{HPF} = I^2R = (0.707I_{max})^2R = (0.5)(I_{max}^2R) = \frac{1}{2}P_{max}$$

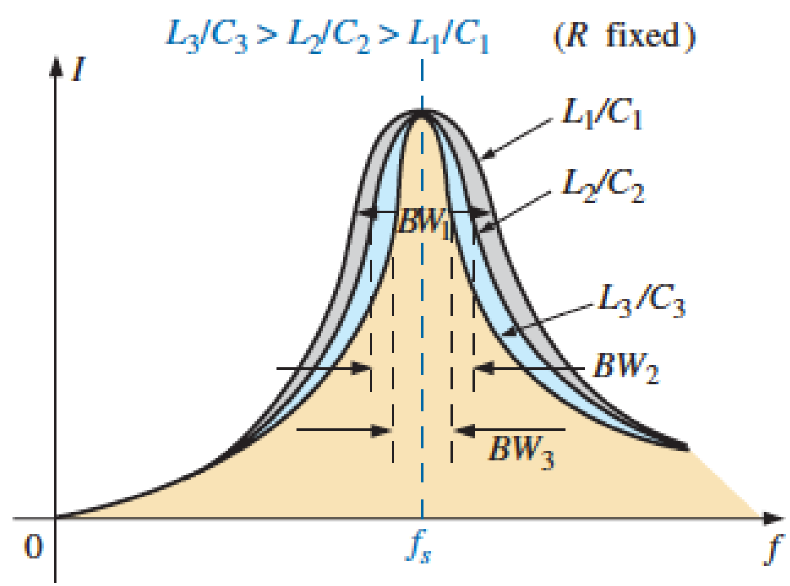
Bandwidth (BW)

If the resistance is made smaller with a fixed inductance and capacitance, the bandwidth decreases. Similarly, if the ratio L/C increases with fixed resistance, the bandwidth again decreases.



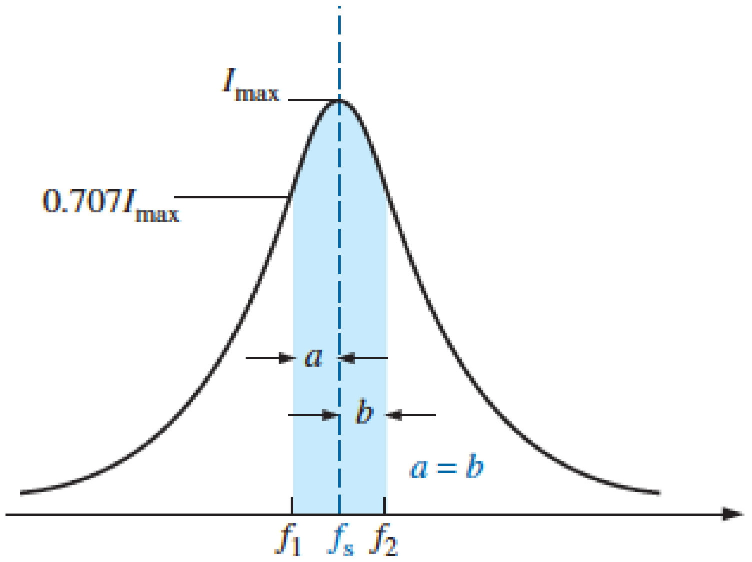
(a)

A small Qs , therefore, is associated with a resonant curve having a large bandwidth, while a large Qs indicates the opposite.



(b)

The cutoff frequencies *f1* and *f2*



$$Z_T = \sqrt{R^2 + (X_L - X_C)^2}$$

becomes

$$\sqrt{2}R = \sqrt{R^2 + (X_L - X_C)^2}$$

or, squaring both sides, that

$$2R^2 = R^2 + (X_L - X_C)^2$$

and,

$$R^2 = (X_L - X_C)^2$$

Taking the square root of both sides gives us

$$R = X_L - X_C \quad \text{or} \quad R - X_L + X_C = 0$$

Let us first consider the case where $X_L > X_C$, which relates to f_2 or ω_2. Substituting $\omega_2 L$ for X_L and $1/\omega_2 C$ for X_C and bringing both quantities to the left of the equal sign, we have

$$R - \omega_2 L + \frac{1}{\omega_2 C} = 0 \quad \text{or} \quad R\omega_2 - \omega_2^2 L + \frac{1}{C} = 0$$

which can be written

$$\omega_2^2 - \frac{R}{L}\omega_2 - \frac{1}{LC} = 0$$

The cutoff frequencies *f1* and *f2*

Solving the quadratic, we have

$$\omega_2 = \frac{-(-R/L) \pm \sqrt{[-(R/L)]^2 - [-(4/LC)]}}{2}$$

and

$$\omega_2 = +\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} + \frac{4}{LC}}$$

$$f_2 = \frac{1}{2\pi}\left[\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \quad \text{(Hz)}$$

$$f_1 = \frac{1}{2\pi}\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}\right] \quad \text{(Hz)}$$

The bandwidth (BW) is

$$BW = f_2 - f_1$$

$$BW = f_2 - f_1 = \frac{R}{2\pi L}$$

$$BW = \frac{R}{2\pi L} = \left(\frac{1}{2\pi}\right)\left(\frac{R}{L}\right) = \left(\frac{f_s}{\omega_s}\right)\left(\frac{\omega_s}{Q_s}\right)$$

$$BW = \frac{f_s}{Q_s}$$

Example

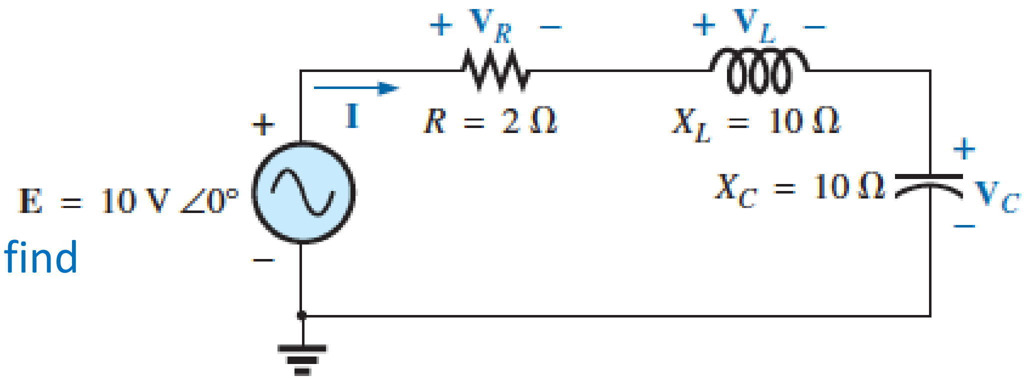
For the series resonant circuit:

a. find I, VR , VL , and VC at resonance?

b. What is the Qs of the circuit?

c. If the resonant frequency is 5000 Hz, find the bandwidth.

d. What is the power dissipated in the circuit at the half-power frequencies?



Solutions:

a. $Z_{T_s} = R = 2\,\Omega$

$$I = \frac{E}{Z_{T_s}} = \frac{10\text{ V}\angle 0°}{2\,\Omega\angle 0°} = \mathbf{5\text{ A}\angle 0°}$$

$$V_R = E = \mathbf{10\text{ V}\angle 0°}$$

$$V_L = (I\angle 0°)(X_L\angle 90°) = (5\text{ A}\angle 0°)(10\,\Omega\angle 90°) = \mathbf{50\text{ V}\angle 90°}$$

$$V_C = (I\angle 0°)(X_C\angle -90°) = (5\text{ A}\angle 0°)(10\,\Omega\angle -90°) = \mathbf{50\text{ V}\angle -90°}$$

b. $Q_s = \dfrac{X_L}{R} = \dfrac{10\,\Omega}{2\,\Omega} = \mathbf{5}$

c. $BW = f_2 - f_1 = \dfrac{f_s}{Q_s} = \dfrac{5000\text{ Hz}}{5} = \mathbf{1000\text{ Hz}}$

d. $P_{HPF} = \dfrac{1}{2}P_{max} = \dfrac{1}{2}I_{max}^2 R = \left(\dfrac{1}{2}\right)(5\text{ A})^2(2\,\Omega) = \mathbf{25\text{ W}}$

Example

The bandwidth of a series resonant circuit is 400 Hz.

a. If the resonant frequency is 4000 Hz, what is the value of Qs?
b. If R = 10 Ω, what is the value of XL at resonance?
c. Find the inductance L and capacitance C of the circuit.

Solutions:

a. $$BW = \frac{f_s}{Q_s} \quad \text{or} \quad Q_s = \frac{f_s}{BW} = \frac{4000\text{ Hz}}{400\text{ Hz}} = \mathbf{10}$$

b. $$Q_s = \frac{X_L}{R} \quad \text{or} \quad X_L = Q_s R = (10)(10\ \Omega) = \mathbf{100\ \Omega}$$

c. $$X_L = 2\pi f_s L \quad \text{or} \quad L = \frac{X_L}{2\pi f_s} = \frac{100\ \Omega}{2\pi(4000\text{ Hz})} = \mathbf{3.98\ mH}$$

$$X_C = \frac{1}{2\pi f_s C} \quad \text{or} \quad C = \frac{1}{2\pi f_s X_C} = \frac{1}{2\pi(4000\text{ Hz})(100\ \Omega)}$$

$$= \mathbf{397.89\ nF}$$

Example

A series R-L-C circuit has a series resonant frequency of 12,000 Hz.

a. If R = 5 Ω, and if XL at resonance is 300 Ω, find the bandwidth.

b. Find the cutoff frequencies.

Solutions:

a. $$Q_s = \frac{X_L}{R} = \frac{300\ \Omega}{5\ \Omega} = 60$$

$$BW = \frac{f_s}{Q_s} = \frac{12{,}000\ \text{Hz}}{60} = \mathbf{200\ Hz}$$

b. Since $Q_s \geq 10$, the bandwidth is bisected by f_s. Therefore,

$$f_2 = f_s + \frac{BW}{2} = 12{,}000\ \text{Hz} + 100\ \text{Hz} = \mathbf{12{,}100\ Hz}$$

and $$f_1 = 12{,}000\ \text{Hz} - 100\ \text{Hz} = \mathbf{11{,}900\ Hz}$$

Example

A series R-L-C circuit is designed to resonate at $w_s = 10^5$ rad/s, have a bandwidth of $0.15f_s$, and draw 16 W from a 120 V source at resonance.

a. Determine the value of R.

b. Find the bandwidth in hertz.

c. Find the nameplate values of L and C.

d. Determine the Qs of the circuit.

Solutions:

a. $$P = \frac{E^2}{R} \quad \text{and} \quad R = \frac{E^2}{P} = \frac{(120\text{ V})^2}{16\text{ W}} = \mathbf{900\ \Omega}$$

b. $$f_s = \frac{\omega_s}{2\pi} = \frac{10^5\text{ rad/s}}{2\pi} = 15{,}915.49\text{ Hz}$$

$$BW = 0.15f_s = 0.15(15{,}915.49\text{ Hz}) = \mathbf{2387.32\ Hz}$$

c. $$BW = \frac{R}{2\pi L} \quad \text{and} \quad L = \frac{R}{2\pi BW} = \frac{900\ \Omega}{2\pi(2387.32\text{ Hz})} = \mathbf{60\ mH}$$

$$f_s = \frac{1}{2\pi\sqrt{LC}} \quad \text{and} \quad C = \frac{1}{4\pi^2 f_s^2 L}$$

$$= \frac{1}{4\pi^2(15{,}915.49\text{ Hz})^2(60 \times 10^{-3})}$$

$$= \mathbf{1.67\ nF}$$

d. $$Q_s = \frac{X_L}{R} = \frac{2\pi f_s L}{R} = \frac{2\pi(15{,}915.49\text{ Hz})(60\text{ mH})}{900\ \Omega} = \mathbf{6.67}$$