

Chapter two Functions

2-1- Exponential and Logarithm functions :

Exponential functions : If a is a positive number and x is any number , we define the exponential function as :

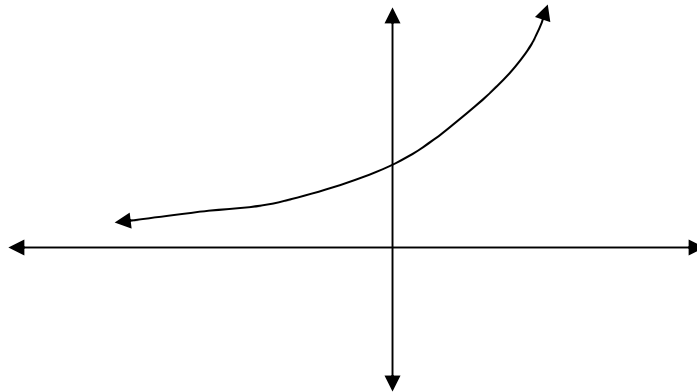
$$y = a^x \quad \text{with domain : } -\infty < x < \infty$$

$$\text{Range : } y > 0$$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x \cdot a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x \cdot y}$.
5. $(a \cdot b)^x = a^x \cdot b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
 $a^\infty = \infty$, $a^{-\infty} = 0$, where $a > 1$.
 $a^\infty = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :

$$y = \log_a x \quad \text{where } x > 0$$

At $a = e = 2.7182828\dots$, we get the natural logarithm and denoted by :

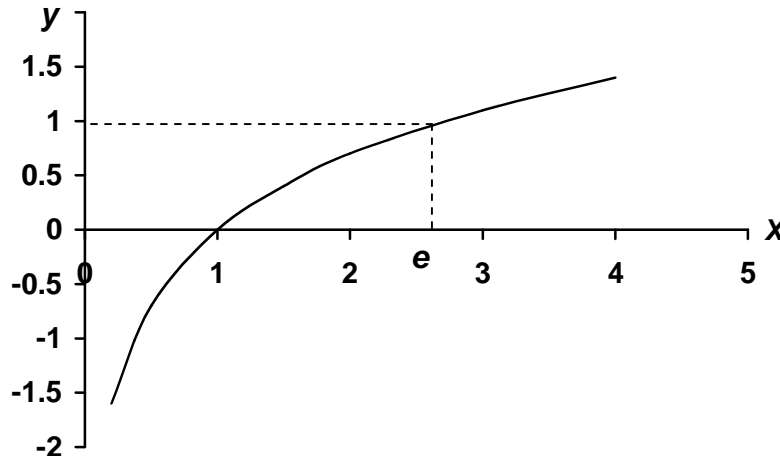
$$y = \ln x$$

Let $x, y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions :

We take Newton's law of cooling :

$$T - T_s = (T_0 - T_s) e^{tk}$$

where T is the temperature of the object at time t .

T_s is the surrounding temperature .

T_0 is the initial temperature of the object .

k is a constant .

EX-1- The temperature of an ingot of metal is $80^\circ C$ and the room temperature is $20^\circ C$. After twenty minutes, it was $70^\circ C$.

- a) What is the temperature will the metal be after 30 minutes?
- b) What is the temperature will the metal be after two hours?
- c) When will the metal be $30^\circ C$?

Sol. :

$$T - T_s = (T_0 - T_s) e^{tk} \Rightarrow 50 = 60 e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

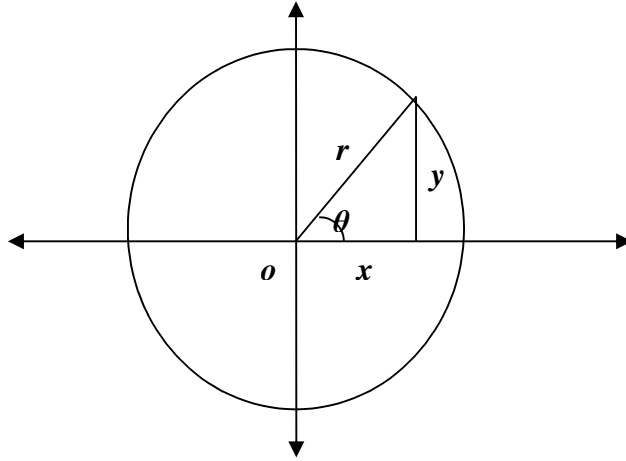
$$a) \quad T - 20 = 60 e^{30(-0.0091)} = 60 * 0.761 = 45.6^\circ C \Rightarrow T = 65.6^\circ C$$

$$b) \quad T - T_s = 60 e^{120(-0.0091)} = 60 * 0.335 = 20.1^\circ C \Rightarrow T = 40.1^\circ C$$

$$c) \quad 10 = 60 e^{-0.0091 t} \Rightarrow -0.0091 t = -\ln 6 \Rightarrow t = 3.3 \text{ hrs.}$$

2-2- Trigonometric functions : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$\sin \theta = \frac{y}{r} = \frac{1}{\csc \theta} \quad , \quad \cos \theta = \frac{x}{r} = \frac{1}{\sec \theta} \quad , \quad \tan \theta = \frac{y}{x} = \frac{1}{\cot \theta} = \frac{\sin \theta}{\cos \theta}$$



The following are some properties of these functions :

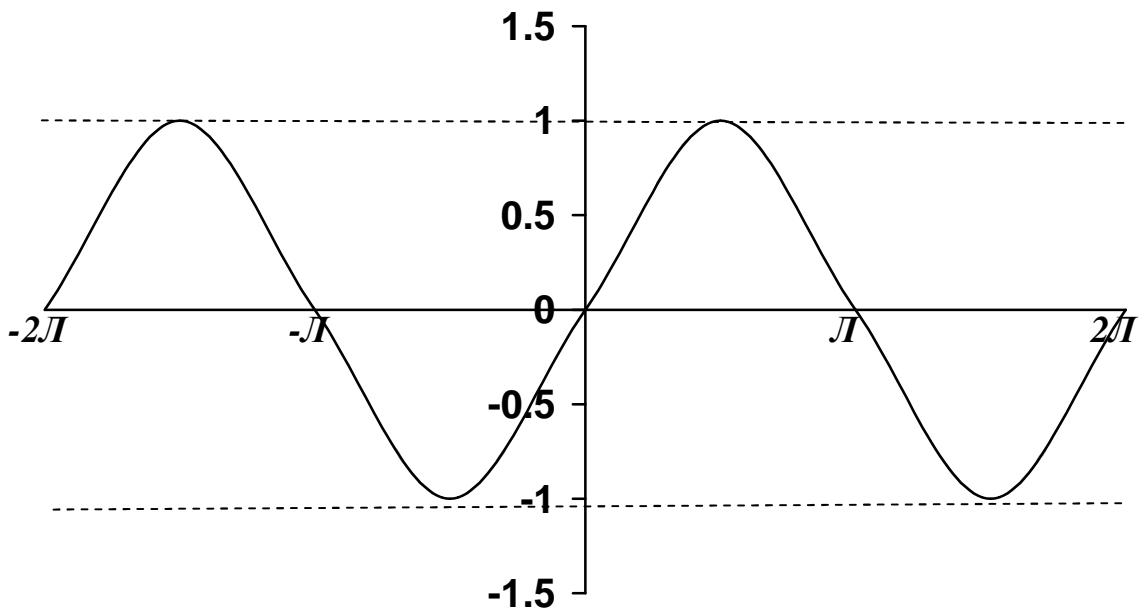
- 1) $\sin^2 \theta + \cos^2 \theta = 1$
- 2) $1 + \tan^2 \theta = \sec^2 \theta$ and $1 + \cot^2 \theta = \csc^2 \theta$
- 3) $\sin(\theta \mp \beta) = \sin \theta \cdot \cos \beta \mp \cos \theta \cdot \sin \beta$
- 4) $\cos(\theta \mp \beta) = \cos \theta \cdot \cos \beta \pm \sin \theta \cdot \sin \beta$
- 5) $\tan(\theta \mp \beta) = \frac{\tan \theta \mp \tan \beta}{1 \pm \tan \theta \cdot \tan \beta}$
- 6) $\sin 2\theta = 2 \sin \theta \cdot \cos \theta$ and $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
- 7) $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ and $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$
- 8) $\sin(\theta \mp \frac{\pi}{2}) = \mp \cos \theta$ and $\cos(\theta \mp \frac{\pi}{2}) = \pm \sin \theta$
- 9) $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$ and $\tan(-\theta) = -\tan \theta$
- 10) $\sin \theta \cdot \sin \beta = \frac{1}{2} [\cos(\theta - \beta) - \cos(\theta + \beta)]$
 $\cos \theta \cdot \cos \beta = \frac{1}{2} [\cos(\theta - \beta) + \cos(\theta + \beta)]$
 $\sin \theta \cdot \cos \beta = \frac{1}{2} [\sin(\theta - \beta) + \sin(\theta + \beta)]$

$$11) \quad \begin{aligned} \sin \theta + \sin \beta &= 2 \sin \frac{\theta + \beta}{2} \cdot \cos \frac{\theta - \beta}{2} \\ \sin \theta - \sin \beta &= 2 \cos \frac{\theta + \beta}{2} \cdot \sin \frac{\theta - \beta}{2} \end{aligned}$$

$$12) \quad \begin{aligned} \cos \theta + \cos \beta &= 2 \cos \frac{\theta + \beta}{2} \cdot \cos \frac{\theta - \beta}{2} \\ \cos \theta - \cos \beta &= -2 \sin \frac{\theta + \beta}{2} \cdot \sin \frac{\theta - \beta}{2} \end{aligned}$$

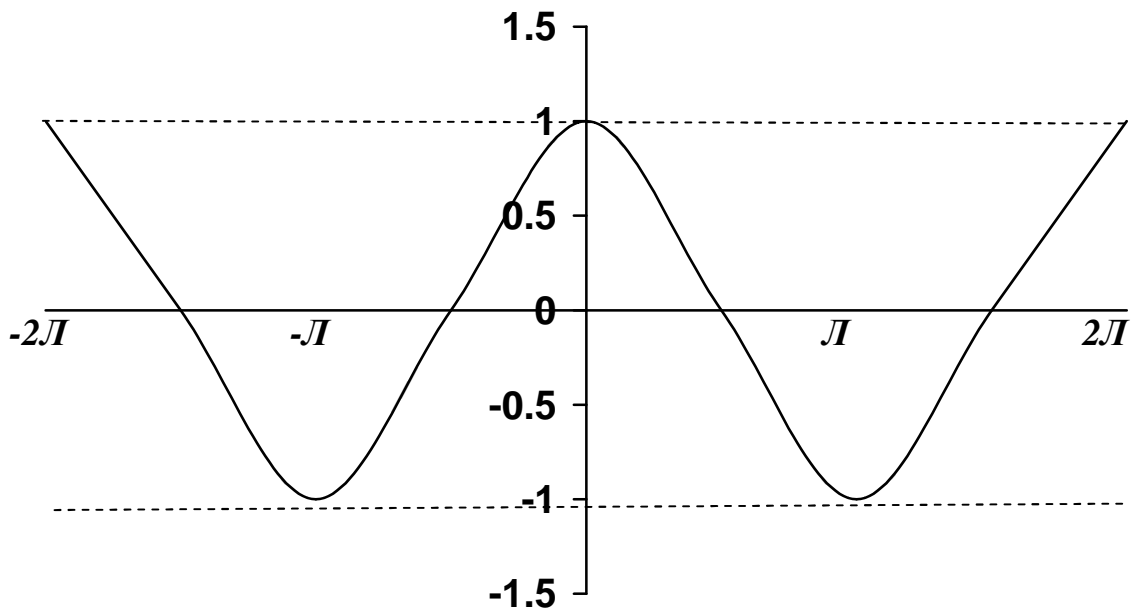
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π
$\sin \theta$	0	1/2	1/√2	√3/2	1	0
$\cos \theta$	1	√3/2	1/√2	1/2	0	-1
$\tan \theta$	0	1/√2	1	√3	∞	0

Graphs of the trigonometric functions are :



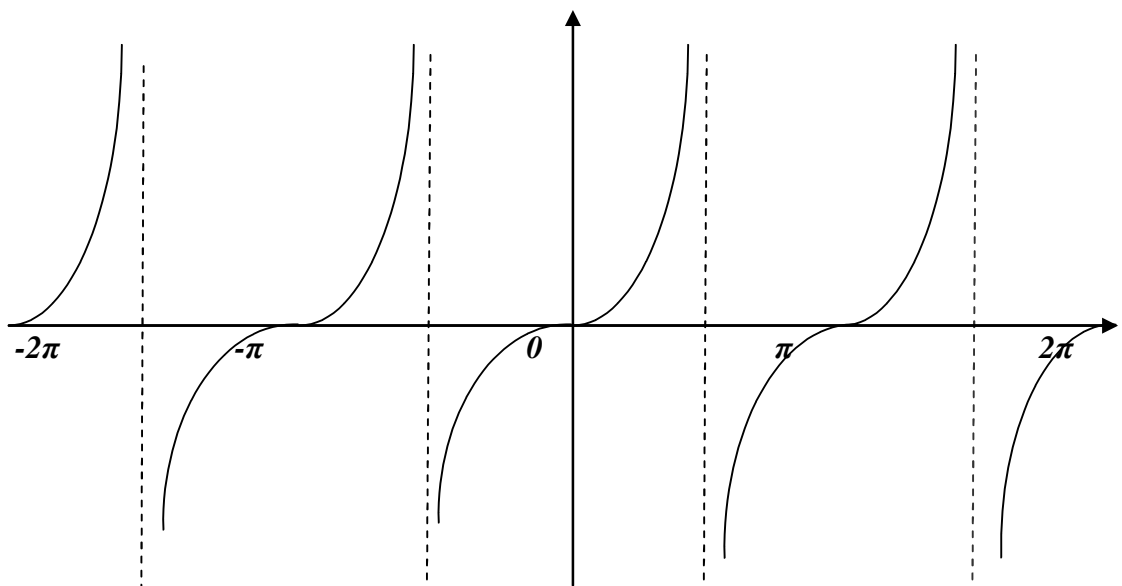
$$y = \sin x \quad D_x : \forall x$$

$$R_y : -1 \leq y \leq 1$$



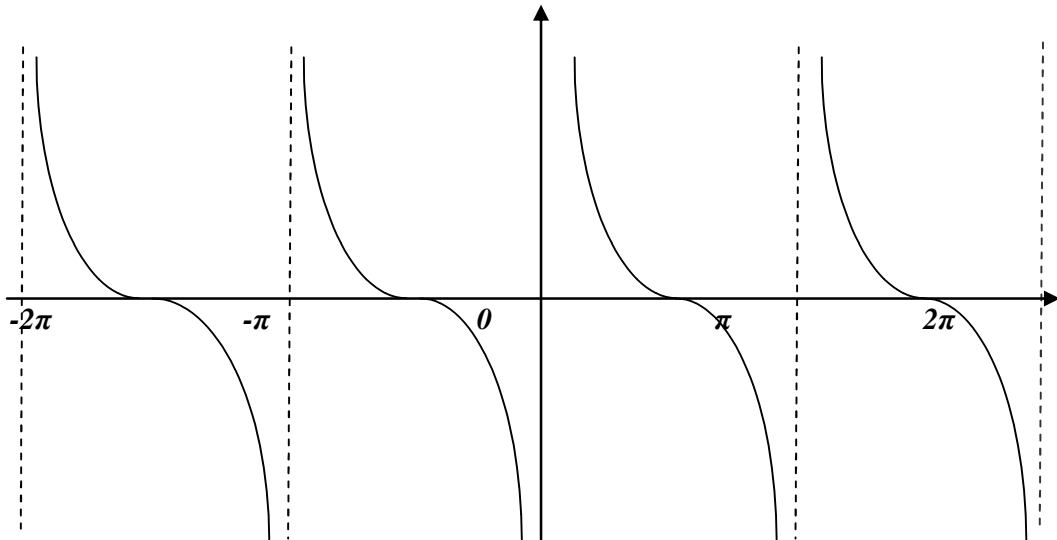
$$y = \cos x \quad D_x : \forall x$$

$$R_y : -1 \leq y \leq 1$$



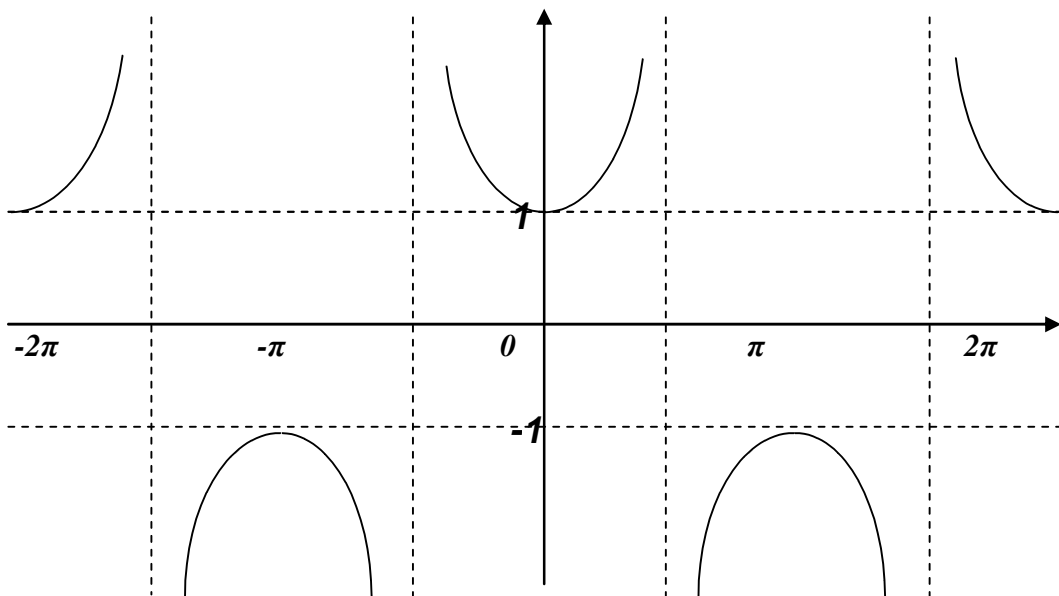
$$y = \tan x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$

$$R_y : \forall y$$



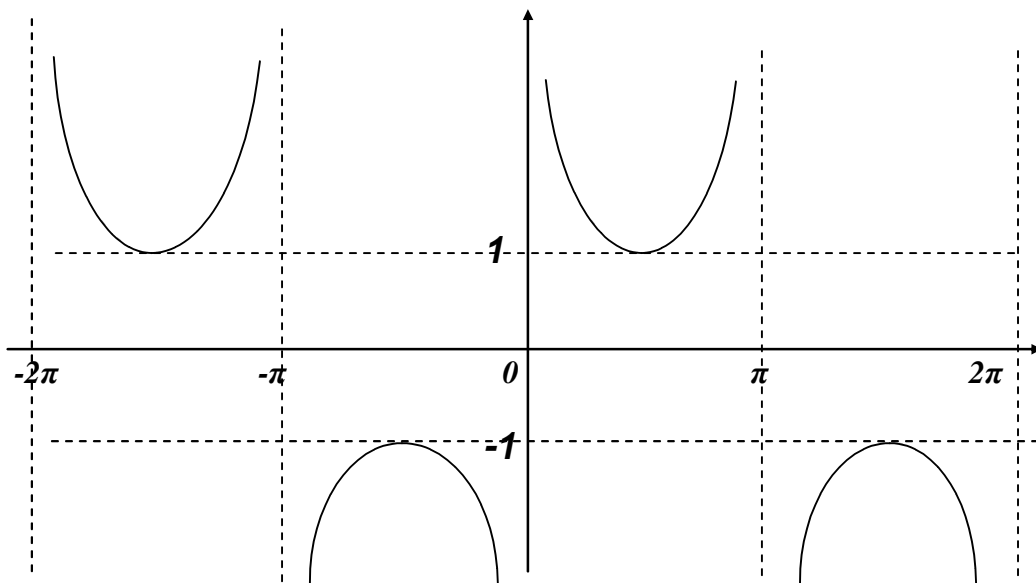
$$y = \text{Cot}x \quad D_x : \forall x \neq n\pi$$

$$R_y : \forall y$$



$$y = \text{Sec}x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$

$$R_y : \forall y \geq 1 \text{ or } y \leq -1$$



$$y = \csc x \quad D_x : \forall x \neq n\pi$$

$$R_y : \forall y \geq 1 \text{ or } y \leq -1$$

Where $n = 0, \pm 1, \pm 2, \pm 3, \dots$

EX-2 - Solve the following equations , for values of θ from 0° to 360° inclusive .

a) $\tan \theta = 2 \sin \theta$ b) $1 + \cos \theta = 2 \sin^2 \theta$

Sol.-

$$a) \quad \tan \theta = 2 \sin \theta \Rightarrow \frac{\sin \theta}{\cos \theta} = 2 \sin \theta$$

$$\Rightarrow \sin \theta (1 - 2 \cos \theta) = 0$$

$$\text{either } \sin \theta = 0 \Rightarrow \theta = 0^\circ, 180^\circ, 360^\circ$$

$$\text{or } \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ, 300^\circ$$

Therefore the required values of θ are $0^\circ, 60^\circ, 180^\circ, 300^\circ, 360^\circ$.

$$b) \quad 1 + \cos \theta = 2 \sin^2 \theta \Rightarrow 1 + \cos \theta = 2(1 - \cos^2 \theta)$$

$$\Rightarrow (2 \cos \theta - 1)(\cos \theta + 1) = 0$$

$$\text{either } \cos \theta = \frac{1}{2} \Rightarrow \theta = 60^\circ, 300^\circ$$

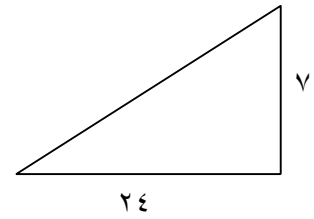
$$\text{or } \cos \theta = -1 \Rightarrow \theta = 180^\circ$$

There the roots of the equation between 0° and 360° are $60^\circ, 180^\circ$ and 300° .

EX-3- If $\tan \theta = 7/24$, find without using tables the values of $\sec \theta$ and $\sin \theta$.
Sol.-

$$\tan \theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$\sec \theta = \frac{r}{x} = \frac{25}{24} \quad \text{and} \quad \sin \theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- Prove the following identities :

- a) $\csc \theta + \tan \theta \cdot \sec \theta = \csc \theta \cdot \sec^2 \theta$
 b) $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$
 c) $\frac{\sec \theta - \csc \theta}{\tan \theta - \cot \theta} = \frac{\tan \theta + \cot \theta}{\sec \theta + \csc \theta}$

Sol.-

a) $L.H.S. = \csc \theta + \tan \theta \cdot \sec \theta = \frac{1}{\sin \theta} + \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta}$
 $= \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cdot \cos^2 \theta} = \frac{1}{\sin \theta} \cdot \frac{1}{\cos^2 \theta} = \csc \theta \cdot \sec^2 \theta = R.H.S.$

b) $L.H.S. = \cos^4 \theta - \sin^4 \theta = (\cos^2 \theta - \sin^2 \theta) \cdot (\cos^2 \theta + \sin^2 \theta)$
 $= \cos^2 \theta - \sin^2 \theta = R.H.S.$

c) $L.H.S. = \frac{\sec \theta - \csc \theta}{\tan \theta - \cot \theta} = \frac{\frac{1}{\cos \theta} - \frac{1}{\sin \theta}}{\frac{\sin \theta}{\cos \theta} - \frac{\cos \theta}{\sin \theta}} = \frac{1}{\sin \theta + \cos \theta}$
 $= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta + \cos \theta} \cdot \frac{\frac{1}{\sin \theta \cdot \cos \theta}}{1} = \frac{\tan \theta + \cot \theta}{\sec \theta + \csc \theta} = R.H.S.$

EX-5- Simplify $\frac{1}{\sqrt{x^2 - a^2}}$ when $x = a \cdot \csc \theta$.

Sol.- $\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2 \csc^2 \theta - a^2}} = \frac{1}{a \sqrt{\cot^2 \theta}} = \frac{1}{a} \tan \theta$.

EX-6- Eliminate θ from the equations :

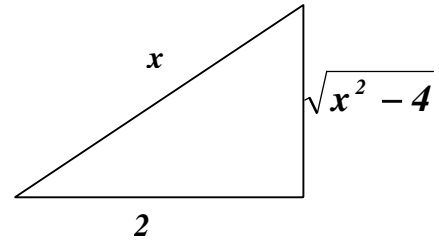
- i) $x = a \sin \theta$ and $y = b \tan \theta$
 ii) $x = 2 \sec \theta$ and $y = \cos 2\theta$

Sol.-

$$i) \quad x = a \cdot \sin \theta \Rightarrow \sin \theta = \frac{x}{a} \Rightarrow \csc \theta = \frac{a}{x}$$

$$y = b \tan \theta \Rightarrow \tan \theta = \frac{y}{b} \Rightarrow \cot \theta = \frac{b}{y}$$

$$\text{Since } \csc^2 \theta = \cot^2 \theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$



$$ii) \quad x = 2 \sec \theta \Rightarrow \cos \theta = \frac{2}{x}$$

$$y = \cos 2\theta \Rightarrow y = \cos^2 \theta - \sin^2 \theta$$

$$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$$

EX-7- If $\tan^2 \theta - 2 \tan^2 \beta = 1$, show that $2 \cos^2 \theta - \cos^2 \beta = 0$.

Sol. -

$$\tan^2 \theta - 2 \tan^2 \beta = 1 \Rightarrow \sec^2 \theta - 1 - 2(\sec^2 \beta - 1) = 1$$

$$\Rightarrow \sec^2 \theta - 2 \sec^2 \beta = 0 \Rightarrow \frac{1}{\cos^2 \theta} - \frac{2}{\cos^2 \beta} = 0$$

$$\Rightarrow 2 \cos^2 \theta - \cos^2 \beta = 0 \quad \text{Q.E.D.}$$

EX-8- If $a \sin \theta = p - b \cos \theta$ and $b \sin \theta = q + a \cos \theta$. Show that :
 $a^2 + b^2 = p^2 + q^2$

Sol. -

$$p = a \cdot \sin \theta + b \cdot \cos \theta \quad \text{and} \quad q = b \cdot \sin \theta - a \cdot \cos \theta$$

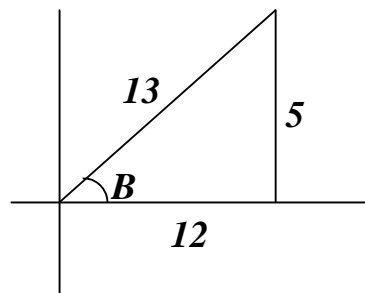
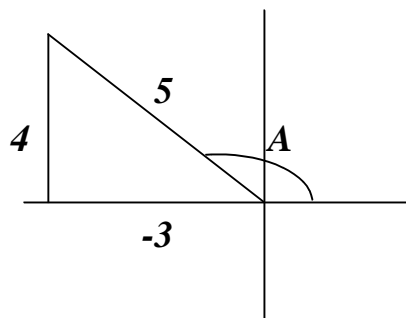
$$p^2 + q^2 = (a \sin \theta + b \cos \theta)^2 + (b \sin \theta - a \cos \theta)^2$$

$$= a^2 (\sin^2 \theta + \cos^2 \theta) + b^2 (\cos^2 \theta + \sin^2 \theta) = a^2 + b^2$$

EX-9- If $\sin A = 4/5$ and $\cos B = 12/13$, where A is obtuse and B is acute. Find, without tables, the values of :

a) $\sin(A - B)$, b) $\tan(A - B)$, c) $\tan(A + B)$.

Sol. -



$$a) \quad \sin(A - B) = \sin A \cdot \cos B - \cos A \cdot \sin B$$

$$= \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}$$

$$b) \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$= \frac{\frac{4}{3} - \frac{5}{12}}{1 + \frac{4}{3} \cdot \frac{5}{12}} = -\frac{63}{16}$$

$$c) \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \cdot \frac{5}{12}} = \frac{33}{56}$$

EX-10 – Prove the following identities:

$$a) \quad \sin(A + B) + \sin(A - B) = 2 \cdot \sin A \cdot \cos B$$

$$b) \quad \tan A + \tan B = \frac{\sin(A + B)}{\cos A \cdot \cos B}$$

$$c) \quad \sec(A + B) = \frac{\sec A \cdot \sec B \cdot \csc A \cdot \csc B}{\csc A \cdot \csc B - \sec A \cdot \sec B}$$

$$d) \quad \frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \cot \theta$$

Sol.-

$$\begin{aligned} a) \quad L.H.S. &= \sin(A+B) + \sin(A-B) \\ &= \sin A \cdot \cos B + \cos A \cdot \sin B + \sin A \cdot \cos B - \cos A \cdot \sin B \\ &= 2 \cdot \sin A \cdot \cos B = R.H.S. \end{aligned}$$

$$\begin{aligned} b) \quad R.H.S. &= \frac{\sin(A+B)}{\cos A \cdot \cos B} = \frac{\sin A \cdot \cos B + \cos A \cdot \sin B}{\cos A \cdot \cos B} \\ &= \tan A + \tan B = L.H.S. \end{aligned}$$

$$\begin{aligned} c) \quad R.H.S &= \frac{\sec A \cdot \sec B \cdot \csc A \cdot \csc B}{\csc A \cdot \csc B - \sec A \cdot \sec B} = \frac{\frac{1}{\cos A} \cdot \frac{1}{\cos B} \cdot \frac{1}{\sin A} \cdot \frac{1}{\sin B}}{\frac{1}{\sin A} \cdot \frac{1}{\sin B} - \frac{1}{\cos A} \cdot \frac{1}{\cos B}} \\ &= \frac{1}{\cos A \cdot \cos B - \sin A \cdot \sin B} = \frac{1}{\cos(A+B)} \\ &= \sec(A+B) = L.H.S. \end{aligned}$$

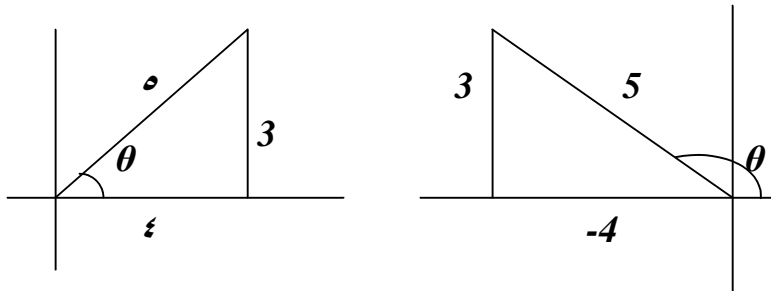
$$\begin{aligned} d) \quad L.H.S. &= \frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \frac{2 \sin \theta \cdot \cos \theta + (\cos^2 \theta - \sin^2 \theta) + 1}{2 \sin \theta \cdot \cos \theta - (\cos^2 \theta - \sin^2 \theta) + 1} \\ &= \frac{2 \sin \theta \cdot \cos \theta + 2 \cos^2 \theta}{2 \sin \theta \cdot \cos \theta + 2 \sin^2 \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta = R.H.S. \end{aligned}$$

EX-11 – Find , without using tables , the values of $\sin 2\theta$ and $\cos 2\theta$, when:

a) $\sin \theta = 3/5$, b) $\cos \theta = 12/13$, c) $\sin \theta = -\sqrt{3}/2$.

Sol. –

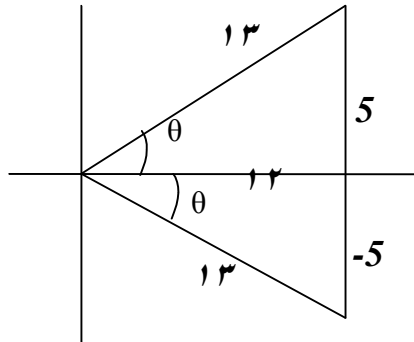
a)



$$\sin 2\theta = 2 \cdot \sin \theta \cdot \cos \theta = 2 \cdot \frac{3}{5} \cdot \left(\pm \frac{4}{5}\right) = \pm \frac{24}{25}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\pm \frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

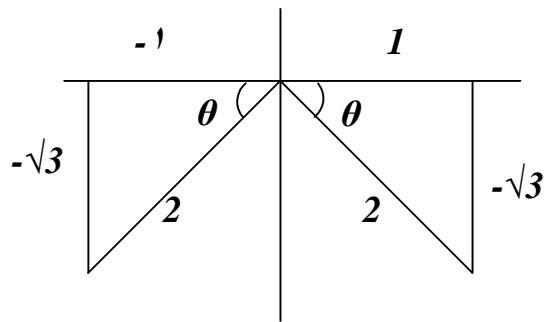
b)



$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\pm \frac{5}{13} \right) \left(\frac{12}{13} \right) = \pm \frac{120}{169}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\frac{12}{13} \right)^2 - \left(\pm \frac{5}{13} \right)^2 = \frac{119}{169}$$

c)



$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(-\frac{\sqrt{3}}{2} \right) \left(\mp \frac{1}{2} \right) = \pm \frac{\sqrt{3}}{2}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(\mp \frac{1}{2} \right)^2 - \left(-\frac{\sqrt{3}}{2} \right)^2 = -\frac{1}{2}$$

EX-12- Solve the following equations for values of θ from 0° to 360° inclusive:

a) $\cos 2\theta + \cos \theta + 1 = 0$, b) $4 \tan \theta \cdot \tan 2\theta = 1$

Sol.-

$$a) \quad \cos 2\theta + \cos \theta + 1 = 0 \Rightarrow 2\cos^2 \theta - 1 + \cos \theta + 1 = 0$$

$$\Rightarrow \cos(2\cos \theta + 1) = 0$$

$$\text{either } \cos \theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ$$

$$\text{or } \cos \theta = -\frac{1}{2} \Rightarrow \theta = 120^\circ, 240^\circ$$

$$\theta = \{90^\circ, 120^\circ, 240^\circ, 270^\circ\}$$

$$b) \quad 4 \cdot \tan \theta \cdot \tan 2\theta = 1 \Rightarrow 4 \cdot \tan \theta \cdot \frac{2 \tan \theta}{1 - \tan^2 \theta} = 1$$

$$\Rightarrow 9 \tan^2 \theta = 1$$

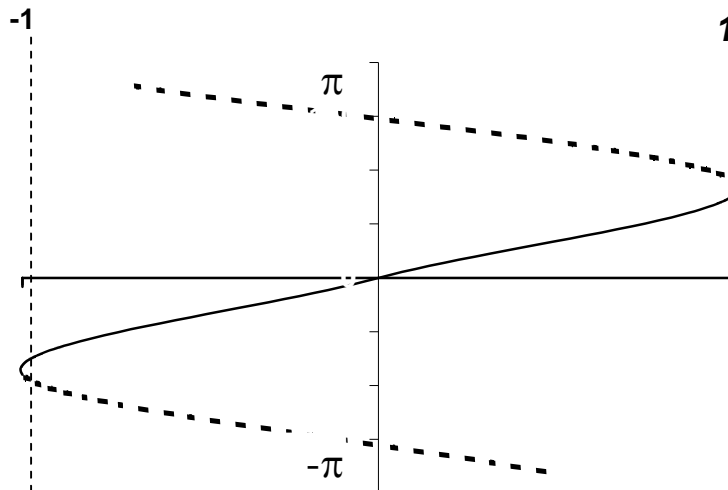
$$\text{either } \tan \theta = \frac{1}{3} \Rightarrow \theta = 18.4^\circ, 198.4^\circ$$

$$\text{or } \tan \theta = -\frac{1}{3} \Rightarrow \theta = 161.6^\circ, 341.6^\circ$$

$$\theta = \{18.4^\circ, 161.6^\circ, 198.4^\circ, 341.6^\circ\}$$

2-3- The inverse trigonometric functions : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :

$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$



$$y = \sin^{-1} x \quad D_x : -1 \leq x \leq 1$$

$$R_y : -90 \leq y \leq 90$$