University of Diyala

College of Engineering

Department of Power and Electrical Machines Engineering

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Second Year – Bachelor (BSc) – EPE 203

Fall Semester

EPE 203 Electric Circuits Analysis I

Lecture 5

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Lecture 5

Energy in A Coupled Circuit and Linear Transformers

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Part 1:

Energy in A Coupled Circuit

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Energy in a Coupled Circuit refers to the total magnetic energy stored in a system of two (or more) inductors that are magnetically linked — meaning the magnetic field produced by the current in one coil links partly or fully with the other coil.

When two inductors are coupled, the total energy is not just the sum of the energies in each inductor separately, because part of the energy is stored in the **mutual magnetic field** shared between them.

Mathematically, for two coupled inductors carrying currents i_1 and i_2 with self-inductances L_1, L_2 and mutual inductance M, the total energy stored in the magnetic field is:

$$\boxed{\omega = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2}$$

The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise.

$$M = k\sqrt{L_1 L_2} \Longrightarrow k = \frac{M}{\sqrt{L_1 L_2}} \qquad k = \frac{\Phi_{12}}{\Phi_1} = \frac{\Phi_{12}}{\Phi_{11} + \Phi_{12}} \qquad k = \frac{\Phi_{21}}{\Phi_2} = \frac{\Phi_{21}}{\Phi_{22} + \Phi_{21}}$$

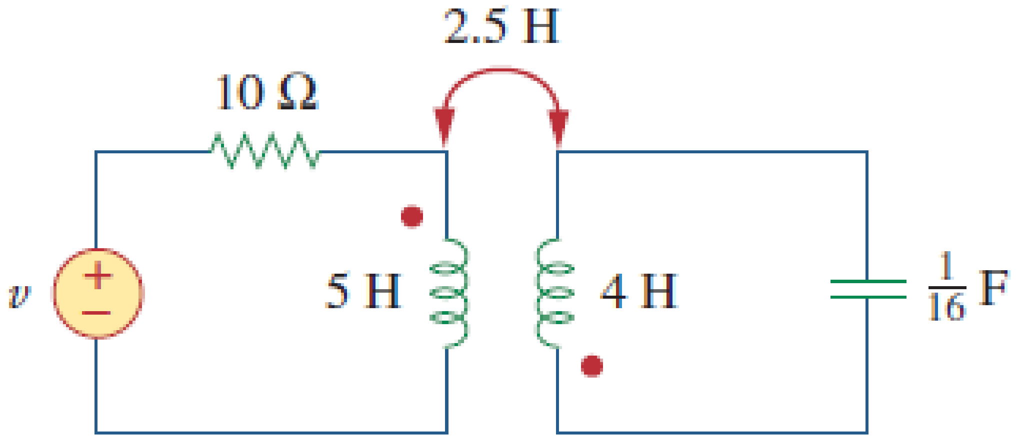
If the entire flux produced by one coil links another coil, then $k = 1$, and we have 100% coupling, or the coils are said to be ***perfectly coupled***. For $k < 0.5$, coils are said to be ***loosely coupled***; and for $k > 0.5$, they are said to be ***tightly coupled***. Thus, The coupling coefficient k is a measure of the magnetic coupling between two coils $0 \le k \le 1$

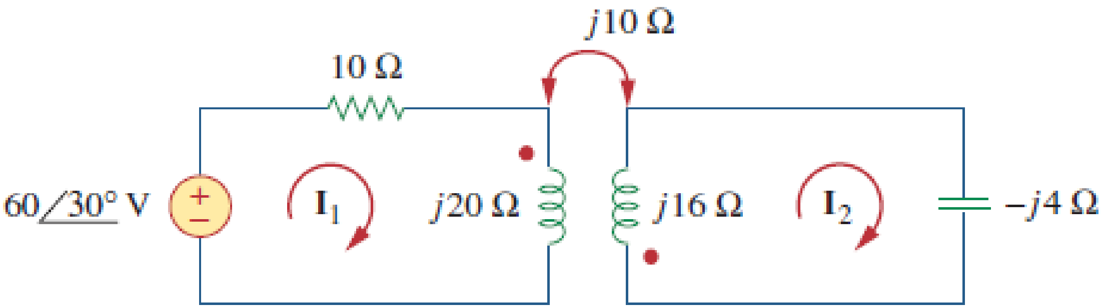
Ex1: For the circuit shown below, determine the coupling coefficient and calculate the energy stored in the coupled inductors at time $t = 1s$ if $v = 60\,cos(4t + 30^{\circ})$ V.

The coupling coefficient is: $k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$

Since $k < 1$ indicating that the inductors are tightly coupled. To find the energy stored, we need to calculate the current. To find the current, we need to obtain the frequency-domain equivalent of the circuit.

$$60\cos(4t + 30^{\circ}) \Rightarrow 60\angle 30^{\circ}, \quad \omega = 4 \text{ rad/s}$$
$$5\text{ H} \Rightarrow j\omega L_1 = j20\ \Omega$$
$$2.5\text{ H} \Rightarrow j\omega M = j10\ \Omega$$
$$4\text{ H} \Rightarrow j\omega L_2 = j16\ \Omega$$
$$\frac{1}{16}\text{ F} \Rightarrow \frac{1}{j\omega C} = -j4\ \Omega$$





For mesh 1, $(10 + j20)I_1 + j10I_2 = 60\angle 30^{\circ}$

For mesh 2, $j10I_1 + (j16 - j4)I_2 = 0 \Longrightarrow I_1 = -1.2I_2$

Substituting this into mesh 1 equation.

$$(10 + j20)(-1.2I_2) + j10I_2 = 60\angle 30^{\circ} \Longrightarrow \mathrm{I}_2 = 3.254\angle 160.6^{\circ}\mathrm{A}$$

$$\mathrm{I}_1 = 3.905\angle -19.4^{\circ}\ \mathrm{A}$$

In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4^{\circ}), \qquad i_2 = 3.254\cos(4t + 160.6^{\circ})$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^{\circ}$, and

$$i_1 = 3.905\cos(229.2^{\circ} - 19.4^{\circ}) = -3.389\ \mathrm{A}$$
$$i_2 = 3.254\cos(229.2^{\circ} + 160.6^{\circ}) = 2.824\ \mathrm{A}$$

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$
$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73\ \mathrm{J}$$

Part 2:

Linear Transformers

A transformer is generally a four-terminal device comprising two (or more) magnetically coupled coils.

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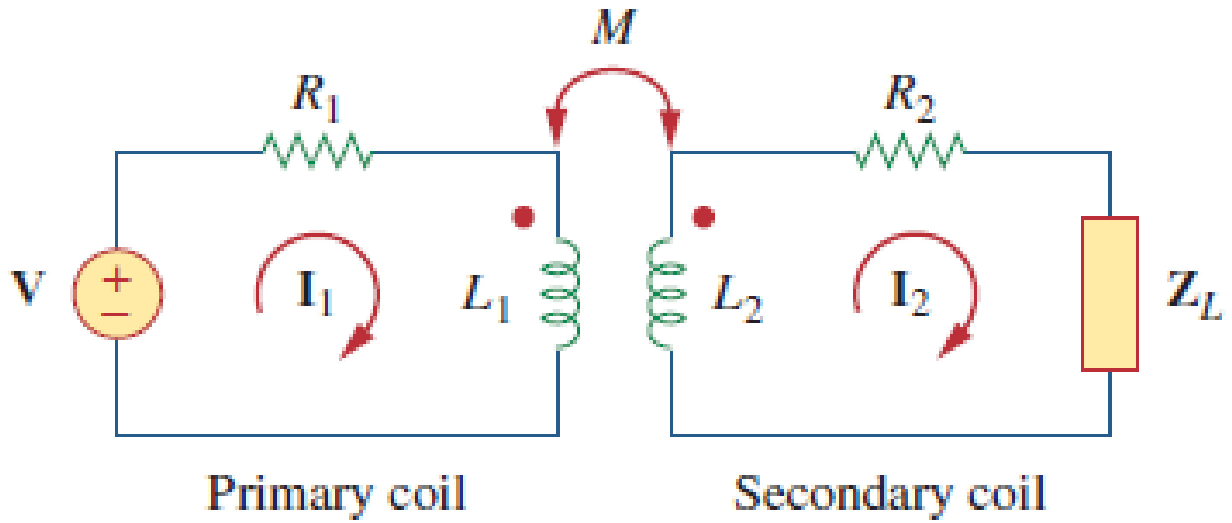
As shown in the figure below, the coil that is directly connected to the voltage source is called the primary winding. The coil connected to the load is called the secondary winding. The resistances R_1 and R_2 are included to account for the losses (power dissipation) in the coils. The transformer is said to be linear if the coils are wound on a magnetically linear material—a material for which the magnetic permeability is constant. Such materials include air, plastic, Bakelite, and wood. In fact, most materials are magnetically linear. Linear transformers are sometimes called air-core transformers, although not all of them are necessarily air-core. They are used in radio and TV sets.



The input impedance Z_{in}

$$Z_{in} = \frac{V}{I_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

Primary Impedance

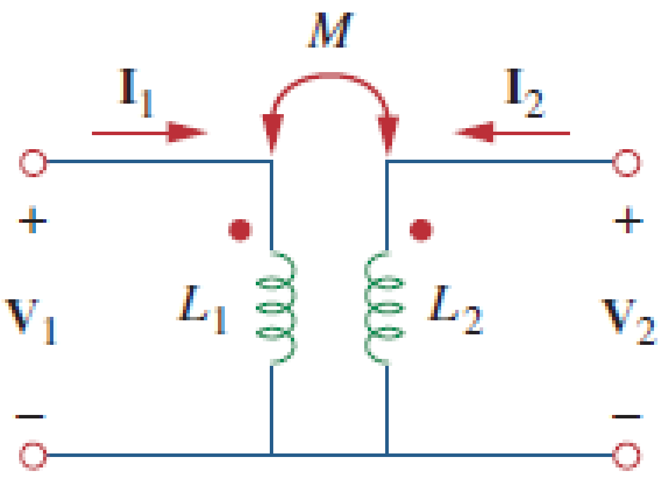
Reflected Impedance Z_R
(due to the coupling between the primary and secondary windings)

The linear transformer shown on the right side shall be replaced by a T & Π equivalent circuits as following.

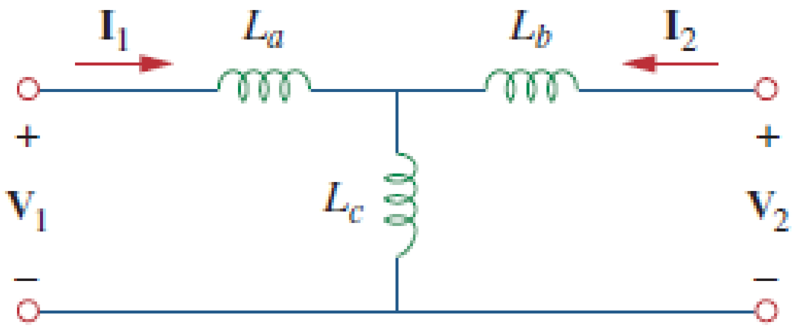
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

By matrix inversion, this can be written as

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$



T equivalent circuit

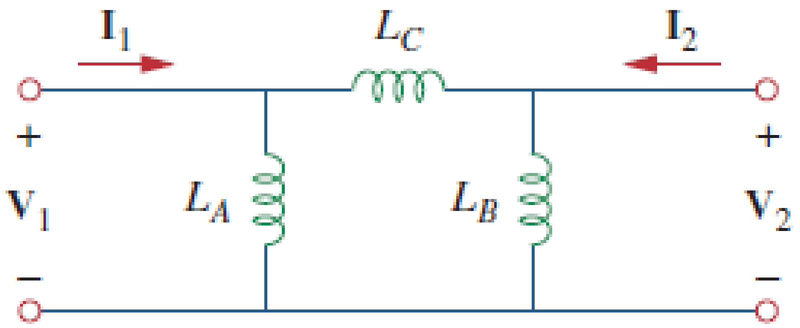


Mesh Analysis

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\boxed{L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M}$$

Π equivalent circuit



Nodal Analysis

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$$\boxed{L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1L_2 - M^2}{L_1 - M} \\ L_C = \frac{L_1L_2 - M^2}{M}}$$

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Ex2: For the circuit shown below, calculate the input impedance and current I_1. If you know that: $Z_1 = 60 - j100\Omega$, $Z_2 = 30 + j40\Omega$, and $Z_L = 80 + j60\Omega$.

$$Z_{in} = \frac{V}{I_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

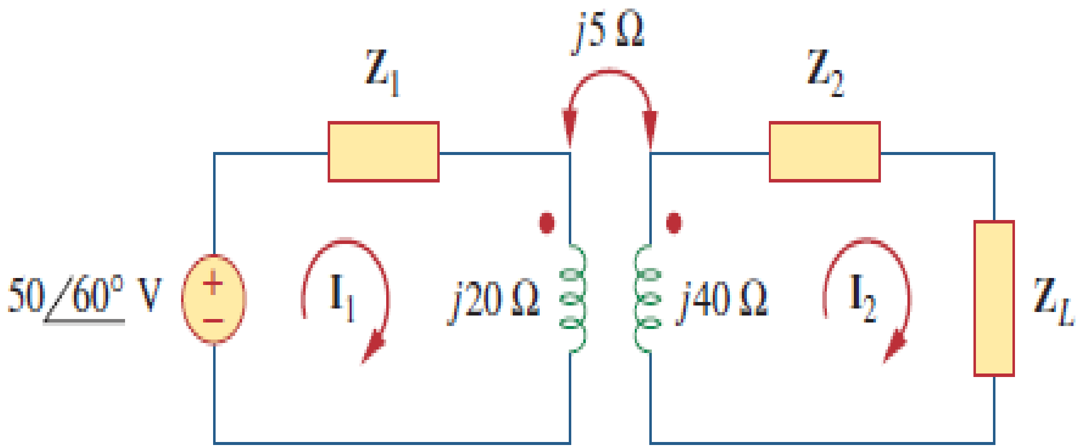
$$\mathbf{Z}_{in} = \mathbf{Z}_1 + j20 + \frac{(5)^2}{j40 + \mathbf{Z}_2 + \mathbf{Z}_L}$$

$$= 60 - j100 + j20 + \frac{25}{110 + j140}$$

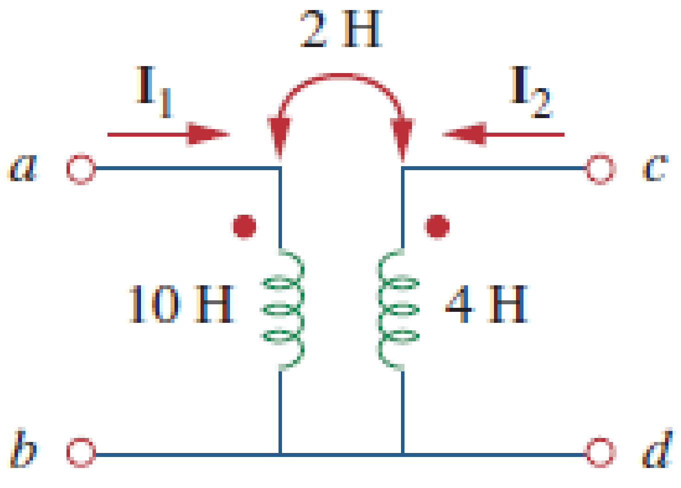
$$= 60 - j80 + 0.14\angle -51.84^\circ$$

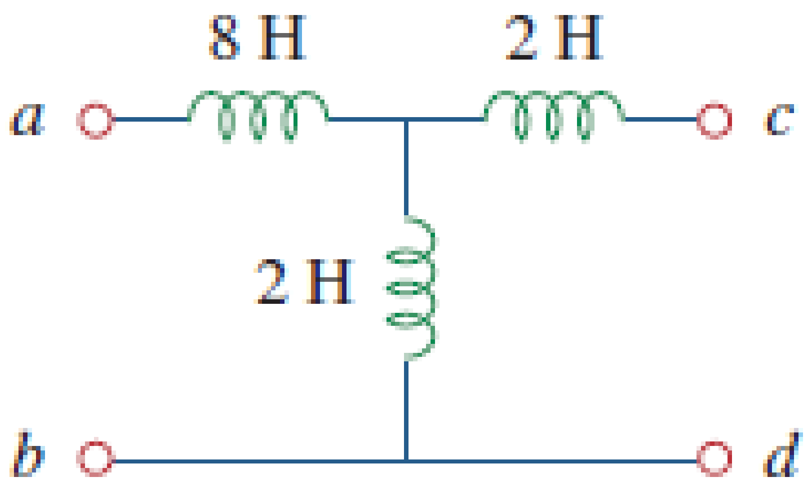
$$= 60.09 - j80.11 = 100.14\angle -53.1^\circ\ \Omega$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_{in}} = \frac{50\angle 60^\circ}{100.14\angle -53.1^\circ} = 0.5\angle 113.1^\circ\ \text{A}$$



Ex3: Determine the T-equivalent circuit of the linear transformer shown below.





Given that $L_1 = 10$, $L_2 = 4$, and $M = 2$, the T-equivalent network has the following parameters:

$$L_a = L_1 - M = 10 - 2 = 8\text{ H}$$

$$L_b = L_2 - M = 4 - 2 = 2\text{ H}, \qquad L_c = M = 2\text{ H}$$

Homework

1) *Practice Problem **13.3** (page 567)*
2) *Practice Problem **13.4** (page 571)*
3) *Practice Problem **13.5** (page 572)*

The practice problems mentioned above, can be found on the textbook of ***"Fundamentals of Electric Circuits"*** 5'th edition by Charles K. Alexander & Matthew N. O. Sadiku