University of Diyala

College of Engineering

Department of Power and Electrical Machines Engineering

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Second Year – Bachelor (BSc) – EPE 204

Spring Semester

EPE 204 Electric Circuits Analysis II

Lecture 6

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Lecture 6

General Second-Order Circuits

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A **second-order circuit** is any circuit that contains: Two energy storage elements: Inductor L and Capacitor C such as RLC series and parallel circuits. These circuits are described by a **second-order differential equation**.

General Differential Equation

The standard form:

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_o x(t) = f(t)$$

$x(t)$: Voltage or Current.

a_1, a_o: Constants depending on R, L, C

$f(t)$: Forcing function (source)

Characteristic Equation

From the homogeneous equation:

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

α: Damping factor (Np/s)

ω_o: Natural frequency (rad/s)

These roots s_1 and s_2 determine the type of response.

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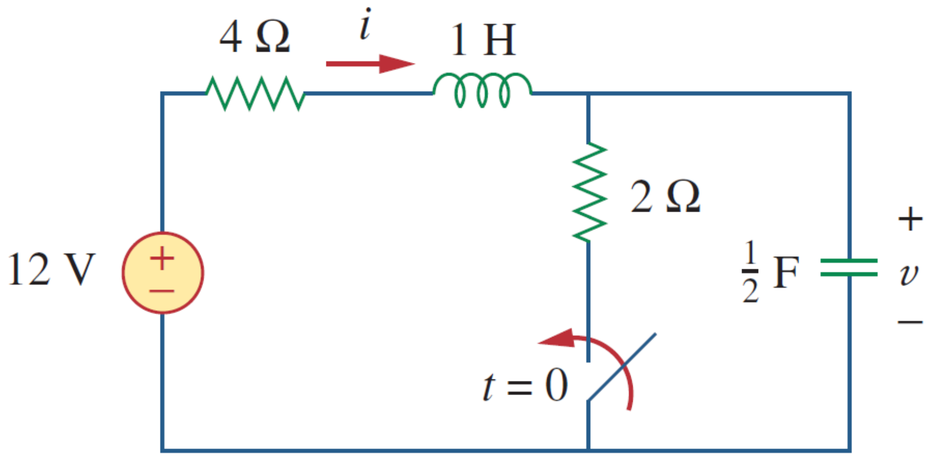
In the second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following procedure:

1) Determine the initial conditions $x(0)$ and $\frac{dx(0)}{dt}$ and the final value $x(\infty)$.
2) Remove the independent sources and find the form of the transient response $x_t(t)$ by applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_t(t)$ with two unknown constants.
3) Find the steady-state response as: $x_{ss}(t) = x(\infty)$. Where $x(\infty)$ is the final value of x, obtained in the first step.
4) Here, the total response is now found as the sum of the transient response and steady-state response: $x(t) = x_t(t) + x_{ss}(t)$
5) Finally determine the constants associated with the transient response by imposing the initial conditions $x(0)$ and $\frac{dx(0)}{dt}$ determined in the first step.

Example 1: Find the complete response for v and i when $t > 0$ in the circuit shown below.



For $t = 0^-$ the switch is open and the circuit is at steady-state, so that the equivalent circuit that used to calculate the initial values is shown in **Fig1**.

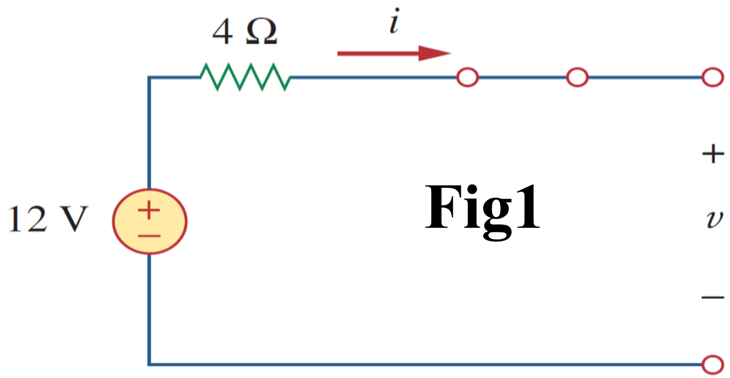


Fig1

$v(0^-) = 12V$ $\qquad$ $i(0^-) = 0A$

For $t = 0^+$ the switch is close and the equivalent circuit that used to calculate the initial values is shown in **Fig2**. Thus, by the continuity of capacitor voltage and inductor current:

$$v(0^+) = v(0^-) = 12V \qquad i(0^+) = i(0^-) = 0A$$

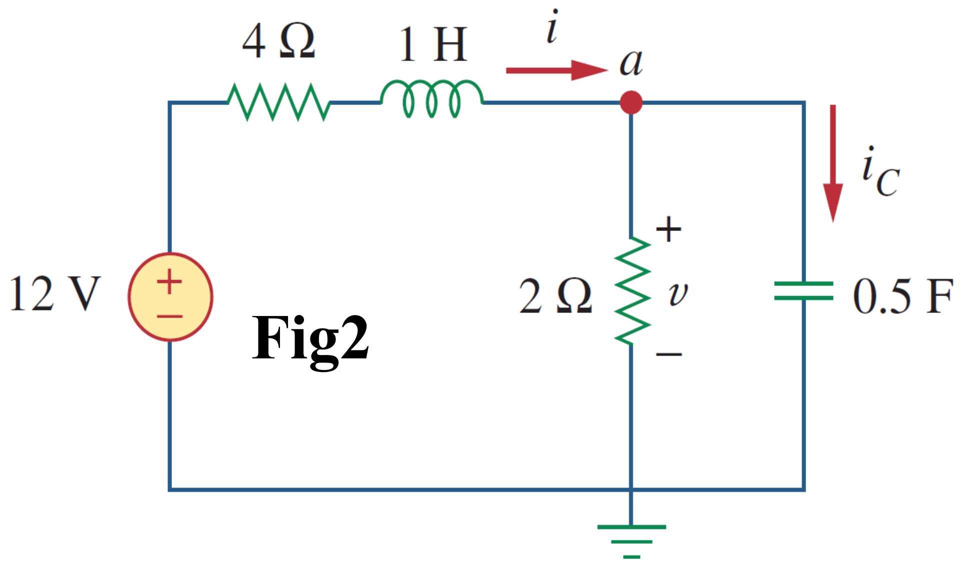


Fig2

$$i_C = C\frac{dv}{dt} \Rightarrow \frac{i_C}{C} = \frac{dv}{dt}$$

Applying KCL at node a in **Fig2**

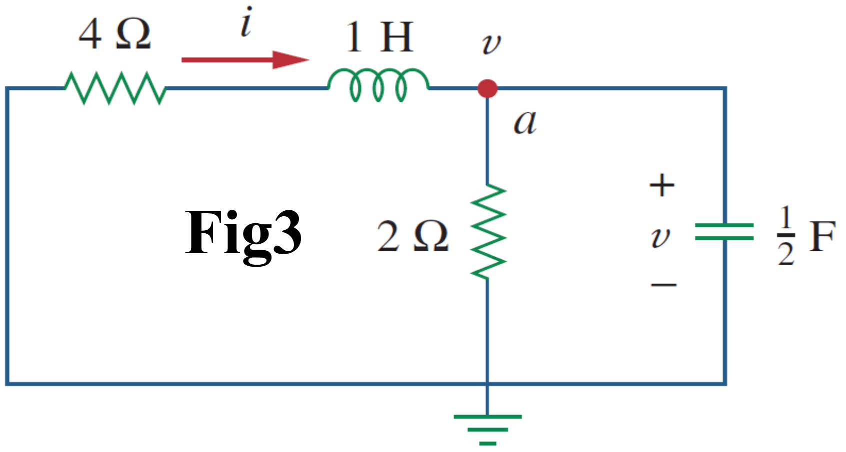
$$i(0^+) = i_c(0^+) + \frac{v(0^+)}{2} \Rightarrow 0 = i_c(0^+) + \frac{12}{2}$$

$$i_c(0^+) = -6\text{A} \Rightarrow \frac{dv}{dt} = \frac{-6}{0.5} = -12\text{V/s} \longrightarrow (1)$$

Now, replace the L with a short circuit and C by an open circuit in **Fig2** to obtain the final values of $i(\infty)$ and $v(\infty)$

$$i(\infty) = \frac{12}{4+2} = 2\text{A} \qquad v(\infty) = Ri(\infty) = 2 \times 2 = 4V$$

Remove the independent sources and find the form of the transient response for $t > 0$, thus we have the circuit of **Fig3.**



Applying KCL at node a in **Fig3.** $$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \rightarrow (2)$$

Applying KVL to the left mesh in **Fig3.** $$4i + 1\frac{di}{dt} + v = 0 \longrightarrow (3)$$

Now, substitute (2) into (3).

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0 \Longrightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0 \longrightarrow (4)$$

From (4), we obtain the characteristic equation as: $s^2 + 5s + 6 = 0 \Longrightarrow s_1 = -2 \,\&\, s_2 = -3$

Thus, the natural response is: $v_n(t) = Ae^{-2t} + Be^{-3t}$

The steady-state response is $v_{ss}(t) = v(\infty) = 4$

The complete response is: $v(t) = v_t(t) + v_{ss}(t) = 4 + Ae^{-2t} + Be^{-3t}$

The complete response is: $v(t) = v_t(t) + v_{ss}(t) = 4 + Ae^{-2t} + Be^{-3t} \longrightarrow (5)$

Now, determine A and B at $t = 0$ using the initial values $v(0^+) = v(0^-) = 12V$

$12 = 4 + A + B \Longrightarrow A + B = 8 \longrightarrow (6)$

Taking the derivative of v for (5) $\Longrightarrow \frac{\mathrm{dv}}{\mathrm{dt}} = -2Ae^{-2t} - 3Be^{-3t} \longrightarrow (7)$

Now, substitute (1) into (7) at $t = 0 \Longrightarrow -12 = -2\mathrm{A} - 3\mathrm{B} \Longrightarrow 2\mathrm{A} + 3\mathrm{B} = 12 \longrightarrow (8)$

From (6) and (8): $A = 12$ and $B = -4$

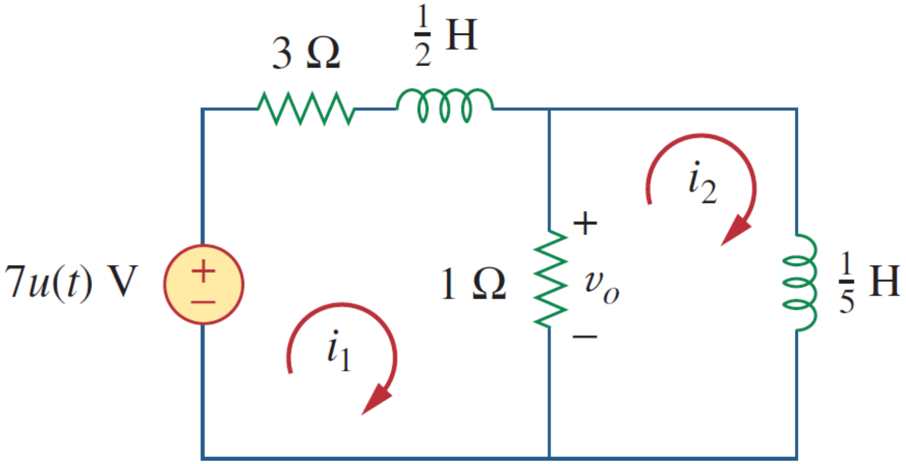
The complete response is: $v(t) = v_t(t) + v_{ss}(t) = 4 + 12e^{-2t} - 4e^{-3t}$ V $t > 0$

From (2)

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t}$$

$i = 2 - 6e^{-2t} + 4e^{-3t}$ A $\qquad t > 0$

Example 2: Find $v_0(t)$ when $t > 0$ in the circuit shown below.



This is a second-order circuit that have two inductors. The first step is to obtain the mesh currents $\boldsymbol{i_1}$ and $\boldsymbol{i_2}$ which happen to be the currents through the inductors. Then, obtain the initial and final values of these currents

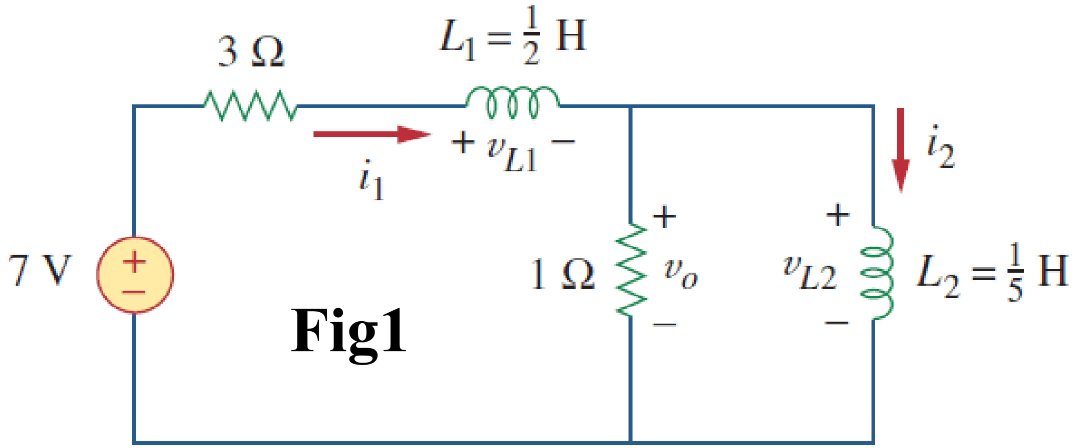


Fig1

For $t < 0$, $7u(t) = 0$ and this leads to $i_1(0^-) = i_2(0^-) = 0$

For $t > 0$, $7u(t) = 7$ and the circuit will be as shown in **Fig1**

Because of the continuity of inductor current,

$i_1(0^-) = i_1(0^+) = 0 \quad i_2(0^-) = i_2(0^+) = 0 \qquad v_0(0^+) = v_{L_2}(0^+) = 1[i_1(0^+) - i_2(0^+)] = 0 \longrightarrow (1)$

Applying KVL to the left loop in **Fig1** at $t = 0^+$

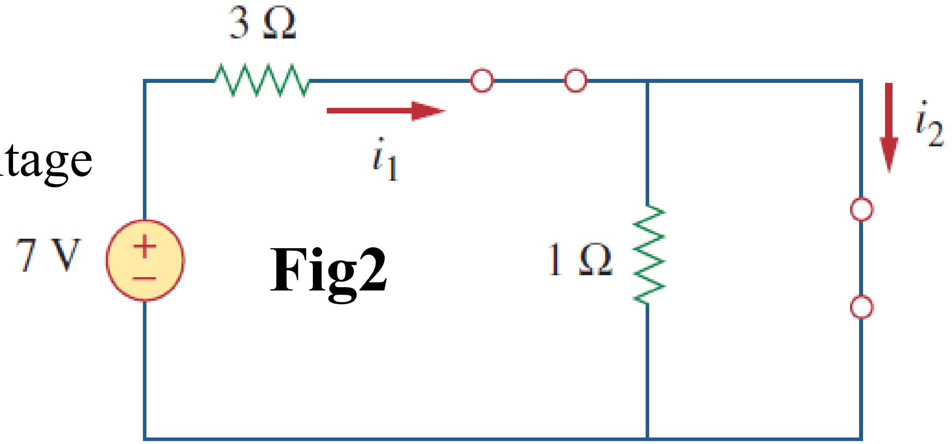
$$7 = 3i_1(0^+) + v_{L_1}(0^+) + v_0(0^+) \Longrightarrow v_{L_1}(0^+) = 7$$

$$v_{L_1}(0^+) = L_1\frac{di_1(0^+)}{dt} \Longrightarrow \frac{di_1(0^+)}{dt} = \frac{v_{L_1}(0^+)}{L_1} = \frac{7}{0.5} = 14\ \mathrm{V/s}$$

$$v_{L_2}(0^+) = L_2\frac{di_2(0^+)}{dt} \Longrightarrow \frac{di_2(0^+)}{dt} = \frac{v_{L_2}(0^+)}{L_2} = 0\ \mathrm{V/s}$$

When $t \longrightarrow \infty$ the circuit reaches steady state, and the inductors become short circuits, as shown in **Fig2**

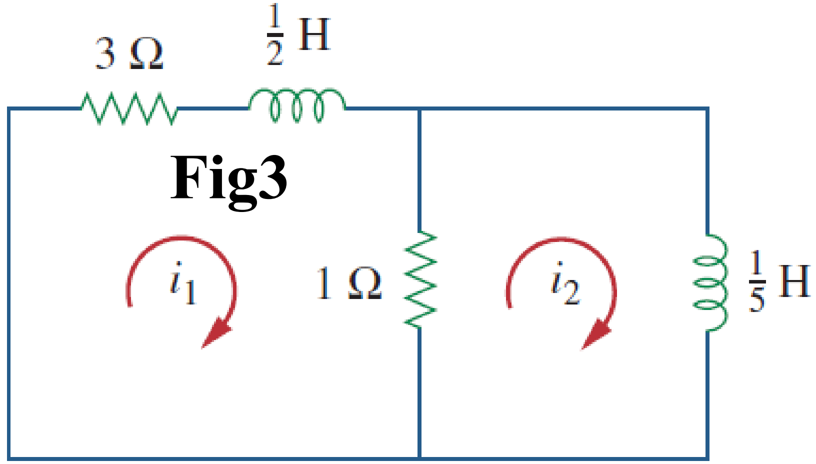
$$i_1(\infty) = i_2(\infty) = \frac{7}{3}\mathrm{A}$$



Now, we obtain the form of the transient responses by removing the voltage source, as shown in **Fig3.** Applying KVL to the two meshes:

$$4i_1 - i_2 + 0.5\frac{d_{i_1}}{dt} = 0 \Longrightarrow i_2 = 4i_1 + 0.5\frac{d_{i_1}}{dt} \longrightarrow (2)$$

$$i_2 + 0.2\frac{d_{i_2}}{dt} - i_1 = 0 \longrightarrow (3)$$



$$4i_1 + 0.5\frac{d_{i_1}}{dt} + 0.8\frac{d_{i_1}}{dt} + 0.1\frac{d^2_{i_1}}{dt} - i_1 = 0$$

$$\frac{d^2_{i_1}}{dt} + 13\frac{d_{i_1}}{dt} + 30i_1 = 0 \Longrightarrow s^2 + 13s + 30 = 0$$

$$s_1 = -3\ \&\ s_2 = -10$$

The form of the transient response is $i_{1n}(t) = Ae^{-3t} + Be^{-10t} \longrightarrow (4)$

The steady-state response is $i_{1ss} = i_1(\infty) = \dfrac{7}{3} \longrightarrow (5)$

From Eqs. (4) and (5), we obtain the complete response as $i_1(t) = \frac{7}{3} + Ae^{-3t} + Be^{-10t} \longrightarrow (6)$

$$\frac{7}{3} + A + B = 0 \longrightarrow (7)$$

Taking the derivative of Eq. (6) and setting $t = 0$ in the derivative and use the Eq. $(*)$

$14 = -3\mathrm{A} - 10\mathrm{B} \longrightarrow (8)$ From Eqs. (7) and (8), $\mathrm{A} = -\frac{4}{3}$ and $\mathrm{B} = -1$

$$i_1(t) = \frac{7}{3} - \frac{4}{3}e^{-3t} - e^{-10t}\,\mathrm{A} \longrightarrow (9) \quad i_2 = 4i_1 + 0.5\frac{d_{i_1}}{dt} \longrightarrow (2) \Longrightarrow i_2 = \frac{7}{3} - \frac{10}{3}e^{-3t} + e^{-10t}\ \mathrm{A}$$

$$v_0(0^+) = v_{L_2}(0^+) = 1[i_1(0^+) - i_2(0^+)] = 0 \longrightarrow (1)$$

$$v_0(0^+) = 2(e^{-3t} - e^{-10t})\mathrm{V}$$