University of Diyala

College of Engineering

Department of Power and Electrical Machines Engineering

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Second Year – Bachelor (BSc) – EPE 204

Spring Semester

EPE 204 Electric Circuits Analysis II

Lecture 7

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Lecture 7

Introduction to the Laplace Transform

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Part 1

Definition of the Laplace Transform

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The **Laplace transform** is an integral transformation of a function ***f*(*t*)** from the time domain into the complex frequency domain, giving ***F*(*s*)**.

Given a function ***f(t)***, its Laplace transform, denoted by ***F(s)*** or $\boldsymbol{\mathcal{L}[f(t)]}$ is defined by:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

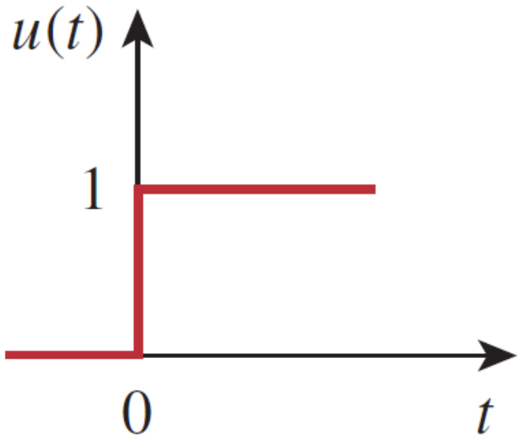
where *s* is a complex variable given by $s = \sigma + j\omega$

The lower limit is specified as 0^- to indicate a time just before $t = 0$. We use 0^- as the lower limit to include the origin and capture any discontinuity of *f*(*t*) at $t = 0$; this will accommodate functions—such as singularity functions— that may be discontinuous at $t = 0$.

Example 1: Determine the Laplace transform of each of the following functions: 1) $u(t)$ 2) $e^{-at}\,u(t), a \geq 0$ 3) $\delta(t)$

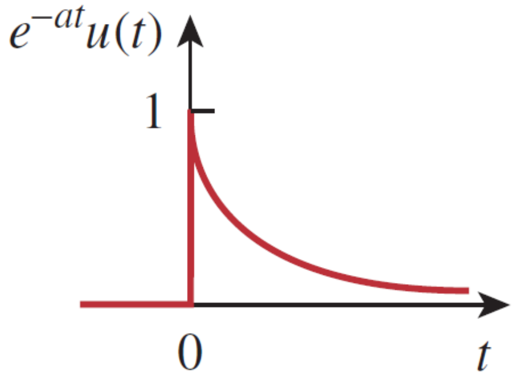
1) For the unit step function $u(t)$,

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$
$$= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$



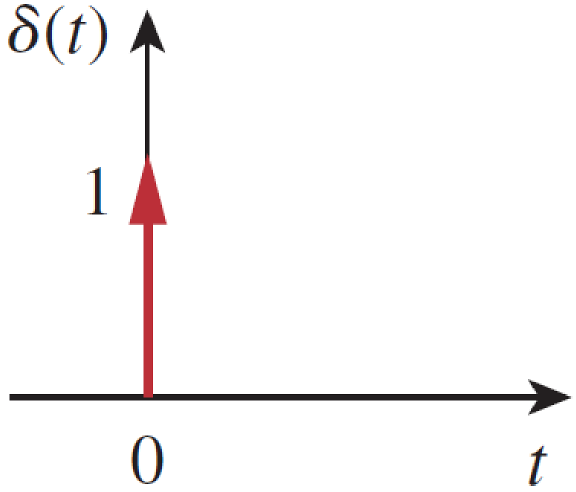
2) For the exponential function,

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$
$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$



3) For the unit impulse function,

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1$$



Properties of the Laplace Transform: TABLE 15.1 Page (687)

Part 2

Circuit Element Models

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Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the *s*-domain.

2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.

3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

Only the first step is new and will be discussed here. As we did in phasor analysis, we transform a circuit in the time domain to the frequency or *s*-domain by Laplace transforming each term in the circuit.

For a resistor, the voltage-current relationship in the time domain is $v(t) = Ri(t)$

Taking the Laplace transform, we get $V(s) = RI(S)$

For an inductor,

$$v(t) = L\frac{d_i(t)}{dt}$$

Taking the Laplace transform of both sides

$$V(s) = L[sI(S) - i(0^-)] = sLI(s) - Li(0^-)$$

or

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}$$

The s-domain equivalents are shown in **Fig. 1**, where the initial condition is modeled as a voltage or current source.

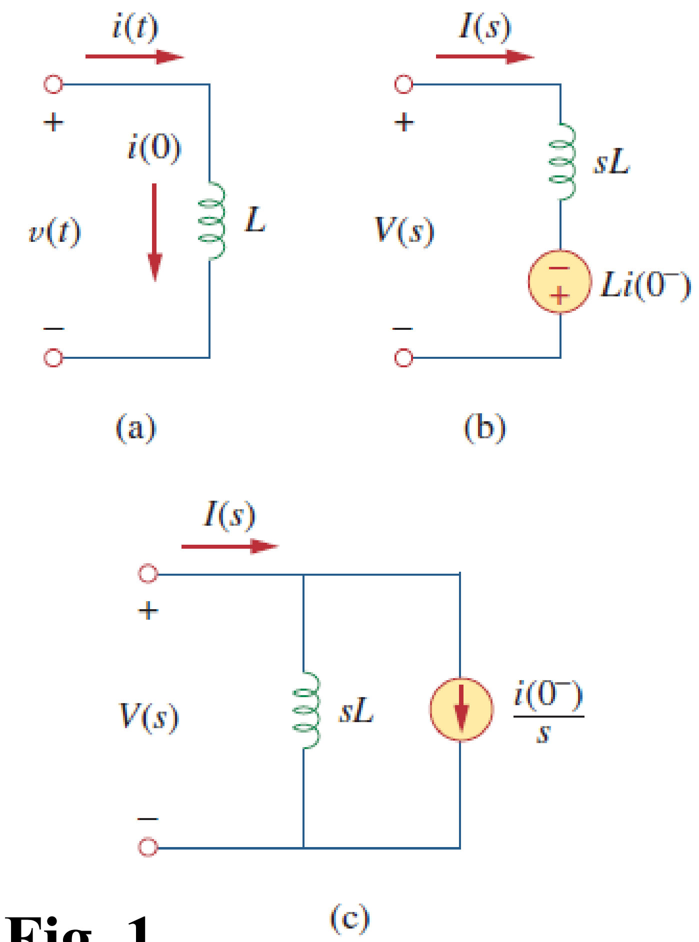

Fig. 1

Representation of an inductor: (a) time-domain, (b,c) s-domain equivalents.

For a capacitor,

$$i(t) = C\frac{d_v(t)}{dt}$$

Taking the Laplace transform of both sides

$$I(s) = C[sV(S) - v(0^-)] = sCV(s) - Cv(0^-) \quad \text{or} \quad V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

The s-domain equivalents are shown in **Fig. 2**. With the s-domain equivalents,

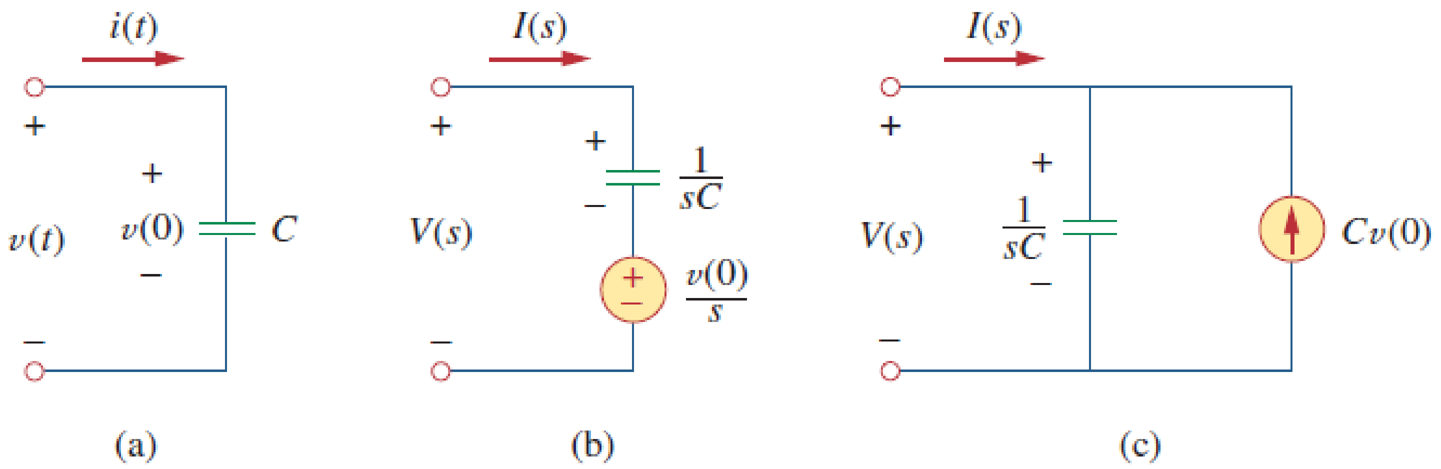

Fig. 2
Representation of a capacitor: (a) time-domain, (b,c) *s*-domain equivalents.

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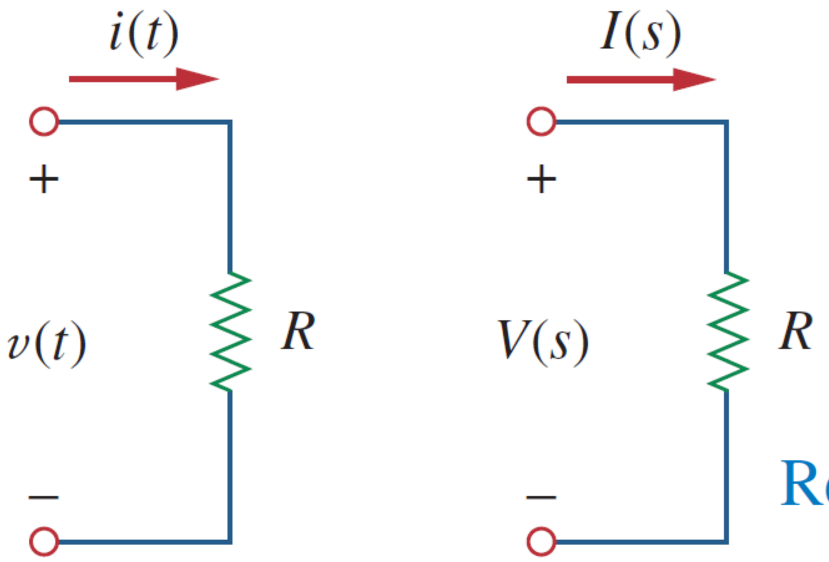
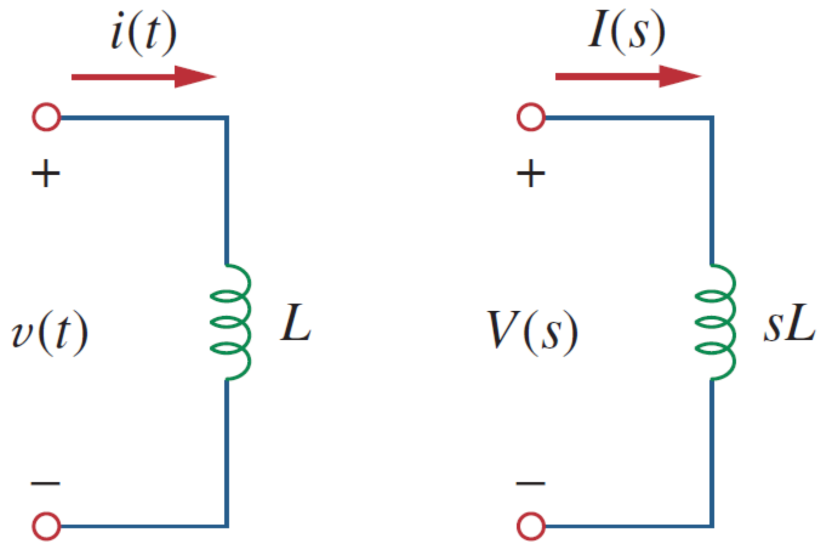
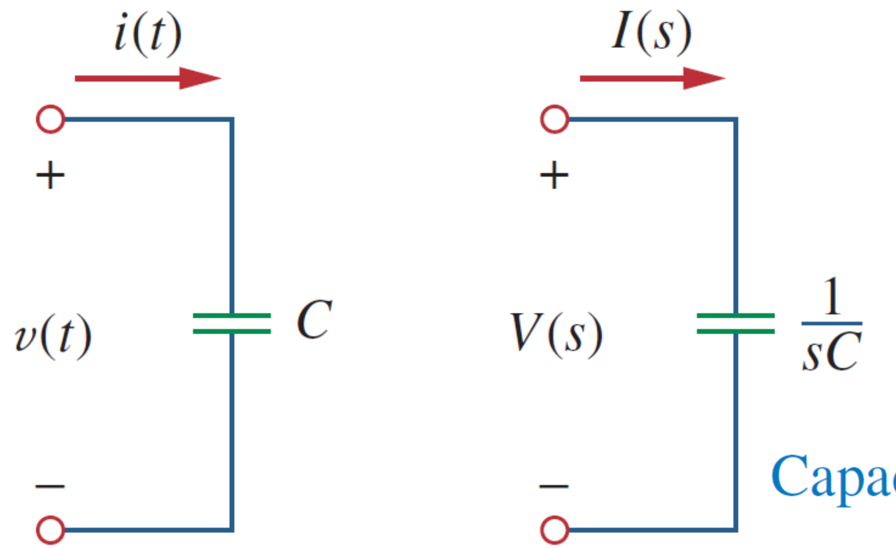


Resistor: $V(s) = RI(s)$



Inductor: $V(s) = sLI(s)$



Capacitor: $V(s) = \dfrac{1}{sC} I(s)$

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Thus, the impedances of the three circuit elements are

$$\text{Resistor:} \quad Z(s) = R$$

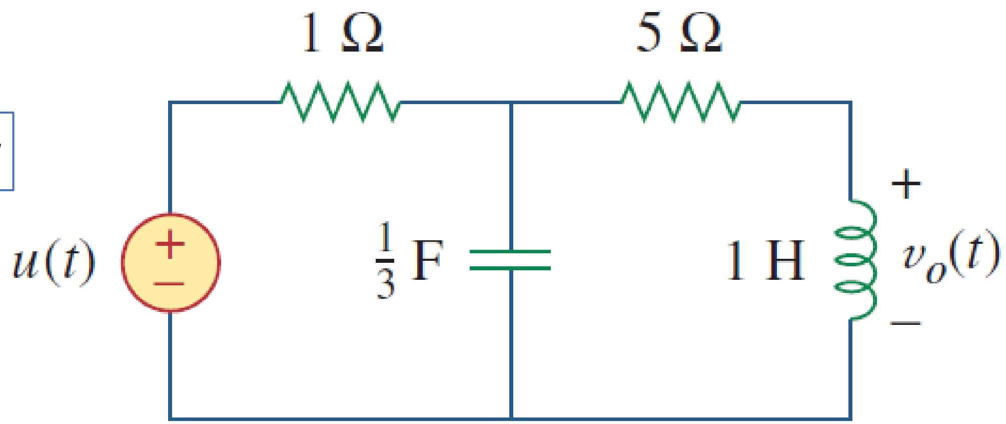
$$\text{Inductor:} \quad Z(s) = sL$$

$$\text{Capacitor:} \quad Z(s) = \frac{1}{sC}$$

The admittance in the s-domain is the reciprocal of the impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$$

Example 1: Find $v_0(t)$ assuming zero initial conditions in the circuit shown below.

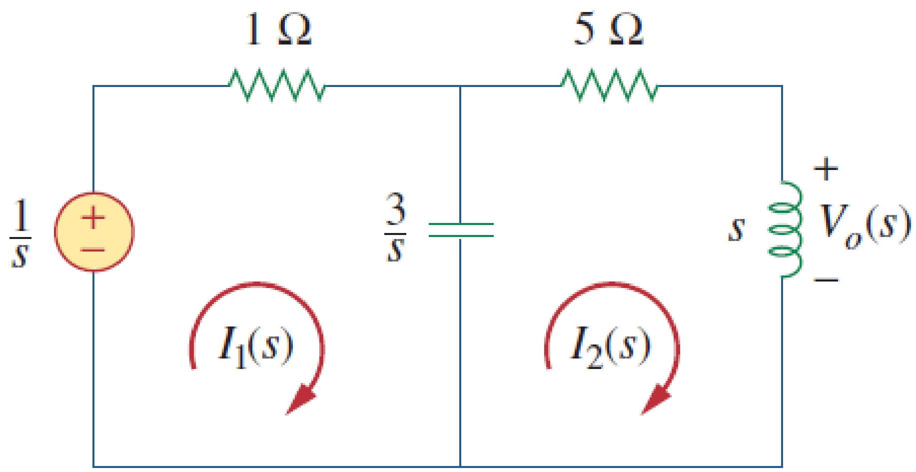


We first transform the circuit from the time domain to the s-domain.

$$u(t) \Rightarrow \frac{1}{s}$$

$$\frac{1}{3}\text{F} \Rightarrow \frac{1}{sC} = \frac{3}{s}$$

$$1H \Rightarrow sL = s$$



For mesh 1: $\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2 \longrightarrow (1)$

For mesh 2: $0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2$

$$\mathrm{I}_1 = \frac{1}{3}(s^2 + 5s + 3)\mathrm{I}_2 \longrightarrow (2)$$

Substituting Eq. (2) into Eq. (1)

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s} I_2$$

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$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \quad \Rightarrow \quad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

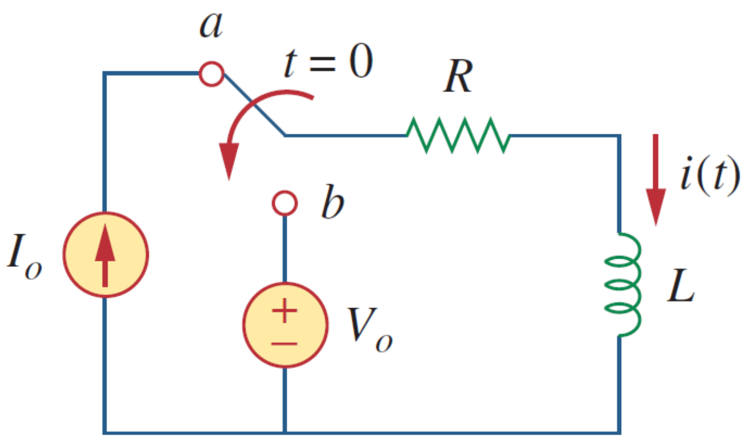
$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s + 4)^2 + (\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin \sqrt{2}t \text{ V}, \qquad t \geq 0$$

Example 2: The switch moves from position a to position b at $t = 0$. Find $i(t)$ for $t > 0$

The initial current through the inductor is $i(0) = I_0$



The circuit transformed to the s-domain **Fig3**. The initial condition is incorporated in the form of a voltage source as $Li(0) = LI_0$. Using mesh analysis,

$$I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0 \quad \text{or}$$

$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} + \frac{V_o/L}{s(s + R/L)}$$

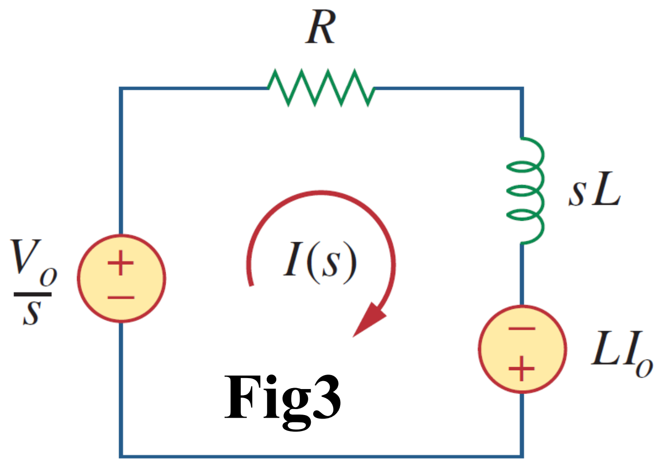


Fig3

Applying partial fraction expansion on the second term on the righthand side

$$I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)}$$

$$\boxed{I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)}}$$

The inverse Laplace transform of this gives

$$i(t) = \underbrace{\left(I_o - \frac{V_o}{R}\right)e^{-t/\tau}}_{\text{Transient response}} + \underbrace{\frac{V_o}{R}}_{\text{Steady-state response}}, \qquad t \geq 0$$