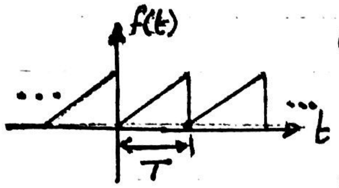
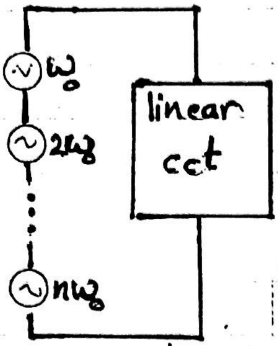
Review of Fourier Series. (For periodic functions)

i) Trigonometric form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

ω_0 = rad. freq. $= \frac{2\pi}{T}$

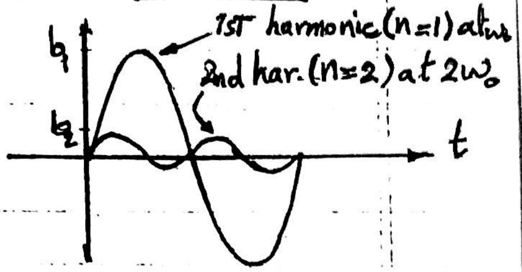




$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t\, dt$$



Notes :-

$$\int_{-T/2}^{T/2} \cos m\omega_0 t \cos n\omega_0 t\, dt = \begin{cases} 0 & m \neq n \\ \frac{T}{2} & m = n \\ T & m = n = 0 \end{cases}$$

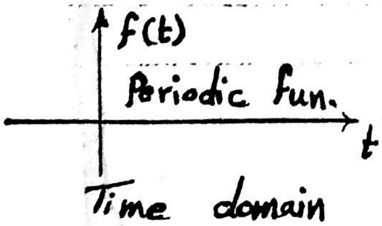
$$\int_{-T/2}^{T/2} \sin m\omega_0 t \sin n\omega_0 t\, dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \cos m\omega_0 t \sin n\omega_0 t\, dt = 0 \quad \text{for all } m \,\&\, n.$$

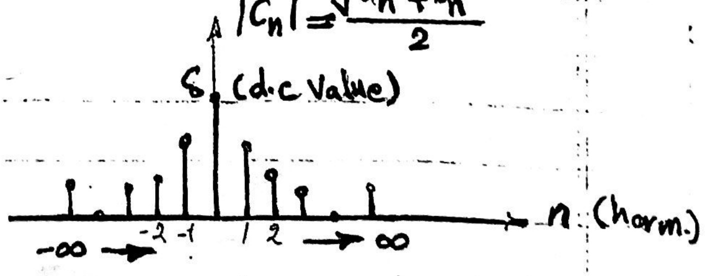
ii) Exponential Form

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt$$



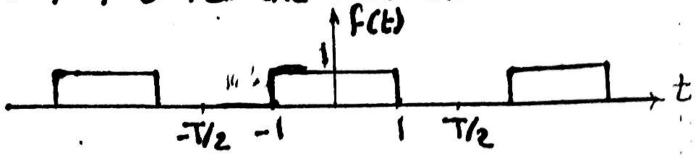
F.T.P



Time domain

Freq. domain (discrete line spectrum)

Ex Find & plot the complex form of F.S for the fun. shown below for $T = 4$, 8 and 16.



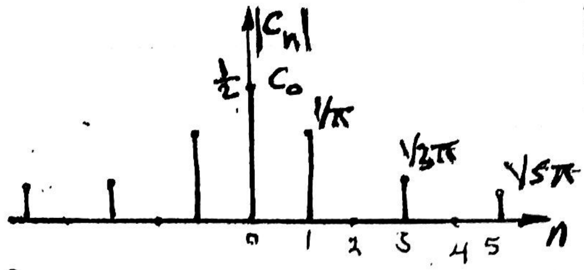
$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_o t}\, dt$$

$$= \frac{1}{T}\int_{-1}^{1} 1\, e^{-jn\omega_o t}\, dt = \frac{1}{T}\, \frac{e^{-jn\omega_o t}}{-jn\omega_o}\Bigg]_{-1}^{1} = \frac{1}{T}\, \frac{e^{jn\omega_o} - e^{-jn\omega_o}}{jn\omega_o} \times \frac{2}{2}$$

$$C_n = \frac{2}{T}\, \frac{\sin n\omega_o}{n\omega_o}$$

i) $T = 4$, $\omega_o = 2\pi/T = \pi/2$

$$C_n = \frac{1}{2}\, \frac{\sin \frac{n\pi}{2}}{\frac{n\pi}{2}}$$



$C_n \to 0$ when $\sin \frac{n\pi}{2} = 0$, i.e

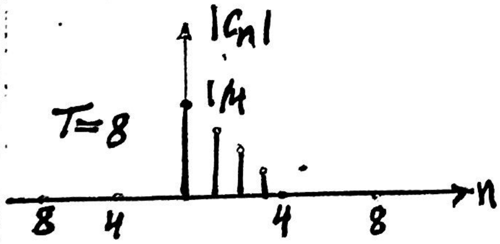
$\frac{n\pi}{2} = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

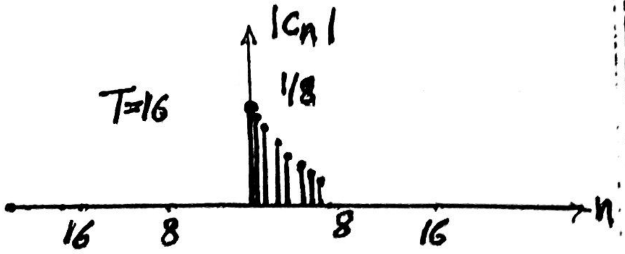
$\therefore n = \pm 2, \pm 4, \pm 6 \ldots$

n	$\lvert C_n \rvert$
1	$1/\pi$
3	$1/3\pi$
5	$1/5\pi$

$$C_o = \frac{1}{2}\, \frac{\frac{\pi}{2}\cos n\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1}{2} \quad (n=0)$$

ii)





As T is increased, the spectrum becomes more dense and as $T \to \infty$, the spectrum becomes continuous rather than discrete.

Fourier Transform (For non-periodic Functions)

Non-periodic functions can be solved using F.T as the period of such functions becomes infinite.

1. F.T.P of non-periodic fun.

$$f(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t} \quad \dots \quad (1)$$

$$C_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t}\, dt \quad \dots \quad (2)$$

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$$f(t)=\sum_{n=-\infty}^{\infty}\left\{\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right\} e^{jn\omega_0 t}$$

as T becomes large $=\frac{2\pi}{\omega_0}$, $n\omega_0=\omega$, $\omega_0=\Delta\omega$

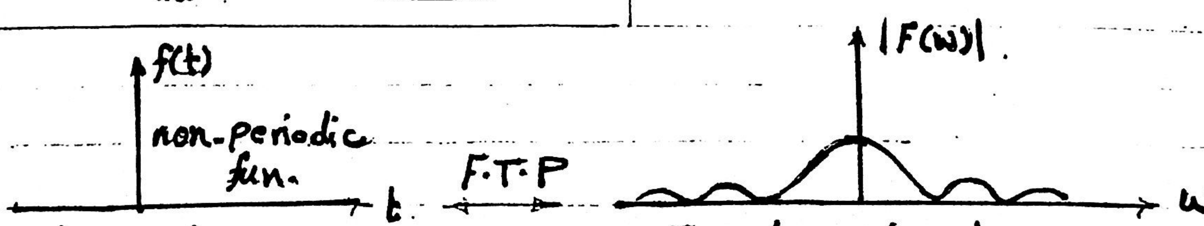
$$f(t)=\sum_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-T/2}^{T/2} f(\tau)\, e^{-j\omega\tau}\, d\tau\right\} e^{j\omega t}\,\Delta\omega$$

as $T\to\infty$, $\Delta\omega\to d\omega$ & $\Sigma\to\int$

$$f(t)=\int_{-\infty}^{\infty}\frac{1}{2\pi}\Big\{\underbrace{\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau}_{F(\omega)}\Big\}\, e^{j\omega t}\, d\omega$$

$$\boxed{f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega} = \text{inverse F.T}$$

$$\boxed{F(\omega)=\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt} = \mathcal{F}\{f(t)\} = \text{direct F.T}$$

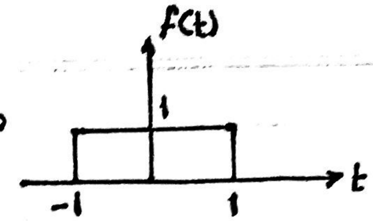


time domain

Freq. domain (continuous line spectrum

4

Plot the spectrum for gate fun. shown below



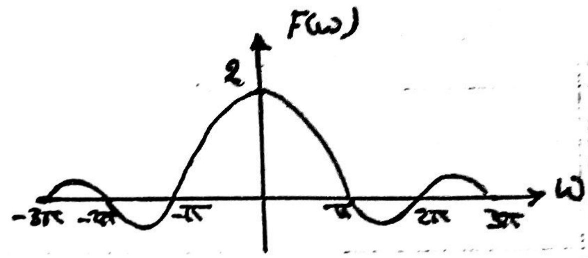
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

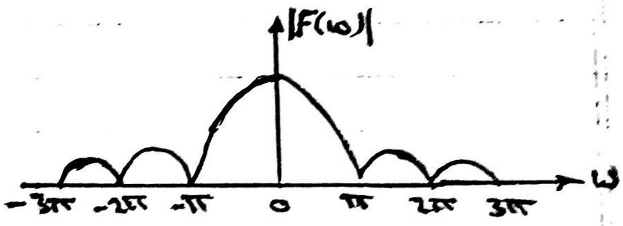
$$= \int_{-1}^{1} 1\, e^{-j\omega t}\, dt = 2\,\frac{\sin\omega}{\omega}$$

$F(\omega) \to 0$ if $\sin\omega = 0$, i.e

$\omega = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

ω	$F(\omega)$
0	2
$\pi/6$	
$\pi/3$	
$\pi/2$	





2- Fourier sine & cosine integrals of non-periodic functions.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \ldots (1)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad \ldots (2)$$

نعوض ② ب 1 ونغير τ المتغير $\nearrow T$

① في ②

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\, \underbrace{e^{-j\omega\tau}\, e^{j\omega t}}_{e^{-j\omega(\tau - t)}}\, d\tau\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ f(\tau)\cos\omega(\tau - t) - \underbrace{J f(\tau)\sin\omega(\tau - t)}_{=0 \text{ since } f(t) \text{ is real.}} \right\} d\tau\, d\omega$$

يستنتج الاستنتاج في الكتاب من 385-386

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F(\omega)\cos\omega t \, d\omega$$ even

$$F(\omega) = 2\int_0^{\infty} f(t)\cos\omega t \, dt = \mathcal{F}\{f(t)\}$$

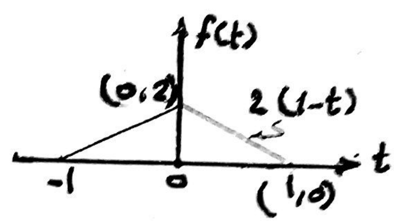
F.T.P of **even** non-periodic fun.

$$f(t) = \frac{1}{\pi}\int_0^{\infty} F(\omega)\sin\omega t \, d\omega$$ odd

$$F(\omega) = 2\int_0^{\infty} f(t)\sin\omega t \, dt = \mathcal{F}\{f(t)\}$$

F.T.P of odd non-Periodic fun.

* Plot the spectrum for the following fun. $F(\omega)$



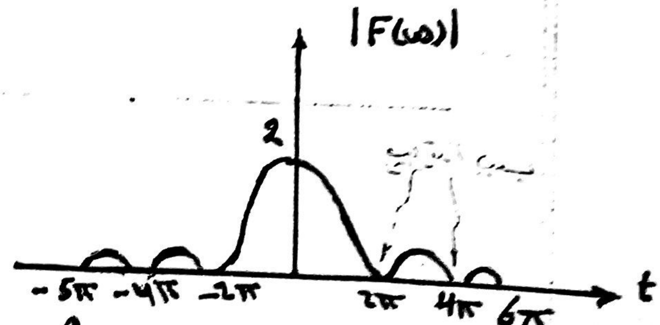
$$F(\omega) = 2\int_0^{\infty} f(t)\cos\omega t \, dt$$ even fun.

$$\boxed{f(t) = 2(1-t)} \qquad 0 \leq t \leq 1$$

$$F(\omega) = 2\int_0^{1} 2(1-t)\cos\omega t \, dt$$

$$= 4\int_0^1 \cos\omega t \, dt - 4\int_0^1 t\cos\omega t \, dt$$

$$F(\omega) = \frac{4}{\omega^2}(1-\cos\omega) \cdot \frac{2}{2}$$



$\because \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$2\theta = \omega \quad , \quad \theta = \frac{\omega}{2}$

$$\therefore F(\omega) = \frac{8}{\omega^2}\sin^2\frac{\omega}{2} = 2\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2$$

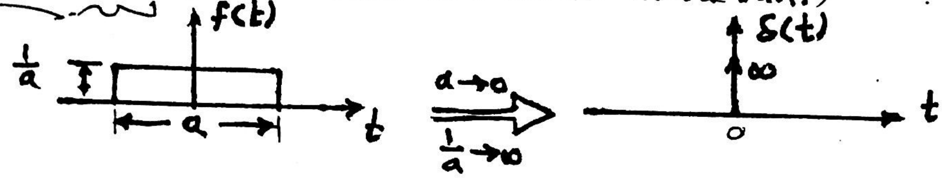
ω	$F(\omega)$
0	2
$\pi/6$	
$\pi/3$	
$\pi/2$	

$F(\omega) = 0$ at $\frac{\omega}{2} = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\omega = \pm 2\pi, \pm 4\pi, \pm 6\pi$

Fourier Transform of some functions:

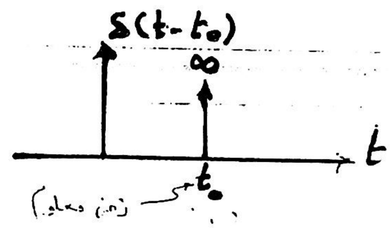
1- Unit impulse Fun. (Delta fun. or Dirac delta fun.)



The area $= a \cdot \frac{1}{a} = 1$, even if $a \to 0$ & $\frac{1}{a} \to \infty$, the area $= 1$
such fun. is called delta fn.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

The area $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases}$$

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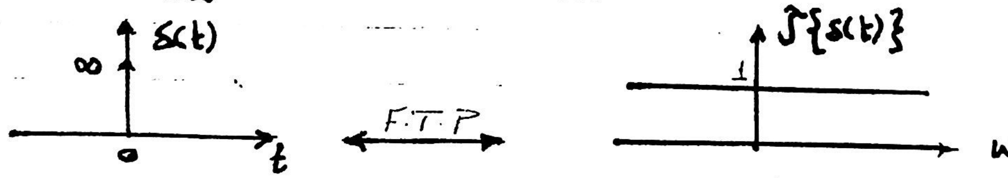
$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

Ex Find $\mathcal{F}\{\delta(t)\}$

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega(0)}\, dt = 1$$



2- Constant (A)

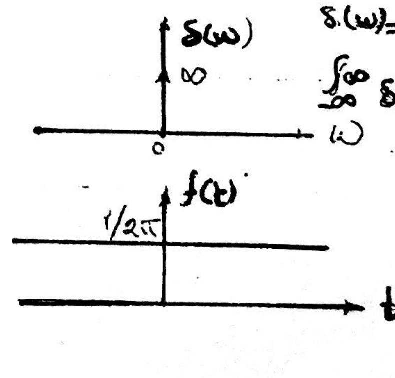
$$\because f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{e^{j(0)t}}{2\pi} \underbrace{\int_{-\infty}^{\infty} \delta(\omega)\, d\omega}_{=1} = \frac{1}{2\pi}$$

$\delta(\omega) = \begin{cases} 0 & \omega \neq 0 \\ \infty & \omega = 0 \end{cases}$

$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$



$$\therefore \mathcal{F}\left\{\frac{1}{2\pi}\right\} = \delta(\omega)$$

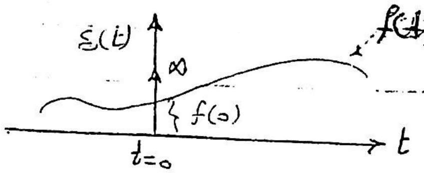
$$\therefore \mathcal{F}\{1\} = 2\pi\, \delta(\omega) = \delta(f)$$

$\mathcal{F}\{A\} = A\,\mathcal{F}\{1\} = A \cdot 2\pi\,\delta(\omega)$

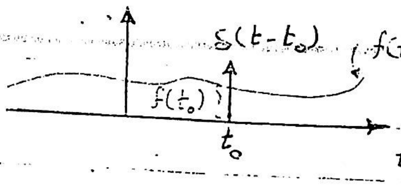
$\therefore\ \mathcal{F}\{A\} = 2\pi A\,\delta(\omega)$

Properties :-

1. $\delta(t) = \delta(-t)$ even fun.
2. $\delta(at) = \dfrac{1}{|a|}\,\delta(t)$
3. $\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$



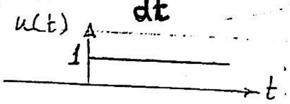
4. $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$

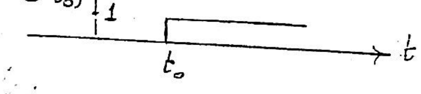


5. Convolution property (تُشرح لاحقاً)

$f(t) \circledast \delta(t-t_0) = f(t-t_0)$

6. $\int_{-\infty}^{\infty} f(t)\,\delta\left[a\,(t-t_0)\right]dt = \dfrac{1}{|a|}\,f(t_0)$
7. $\delta(t) = \dfrac{d}{dt}u(t)$ & $\delta(t-t_0) = \dfrac{d}{dt}u(t-t_0)$





Ex Evaluate $\int_{-\infty}^{\infty} \delta(1-\pi t)\cos(1/t)\,dt$

$\int_{-\infty}^{\infty} \delta\left[\overset{a}{-\pi}\left(t - \overset{t_0}{\frac{1}{\pi}}\right)\right] \cos(1/t)\,dt$ باستخدام الخاصية رقم 6 بعد الترتيب

$= \dfrac{1}{|-\pi|}\cos\left(\dfrac{1}{1/\pi}\right) = -\dfrac{1}{\pi}$

3- Exponential Fun. ($e^{j w_o t}$)

To find fourier transform for the exp. fn. we can benefit from the forth properity of delta (δ) fn. properities

$$f(w_o) = \int_{-\infty}^{\infty} F(w)\,\delta(w-w_o)\,dw$$

$$e^{j w_o t} = \int_{-\infty}^{\infty} \delta(w-w_o)\,e^{j w t}\,dW$$

$$e^{j w_o t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(w-w_o)\,e^{j w t}\,dW$$

$\therefore$ according to difinition :

$$\boxed{\mathcal{F}\{e^{j w_o t}\} = 2\pi\,\delta(w-w_o)} \xrightarrow[W_o=0]{\text{if}} \boxed{\mathcal{F}\{1\} = 2\pi\,\delta(w)}$$

4- Cos(wot) & Sin(wot)

$$\mathcal{F}\{\cos w_o t\} = \mathcal{F}\left\{\frac{e^{j w_o t} + e^{-j w_o t}}{2}\right\}$$

$$= \frac{1}{2}\left[2\pi\,\delta(w-w_o) + 2\pi\,\delta(w+w_o)\right]$$

$$= \pi\left[\delta(w-w_o) + \delta(w+w_o)\right]$$

in the same way

$$\mathcal{F}\{\sin w_o t\} = -j\pi\left[\delta(w-w_o) - \delta(w+w_o)\right]$$

4- Positive time exponential

$$\mathcal{F}\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

$$\therefore \mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$$

in the same way $\mathcal{F}\{e^{at}u(-t)\} = \frac{1}{a-j\omega}$

5- Signum function

$$\operatorname{Sgn}(t) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

To find its fouriar transform we can rewrite it as follow

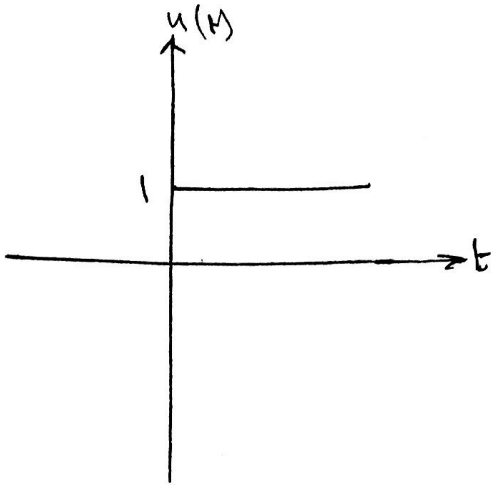
$$\operatorname{Sgn}(t) = \lim_{a \to 0} \left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$\therefore \mathcal{F}\{\operatorname{sgn}(t)\} = \lim_{a \to 0} \left[\mathcal{F}\{e^{-at}u(t)\} - \mathcal{F}\{e^{at}u(-t)\}\right]$$

$$= \lim_{a \to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right]$$

$$= \frac{2}{j\omega}$$

6- Unit Step $u(t)$



$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

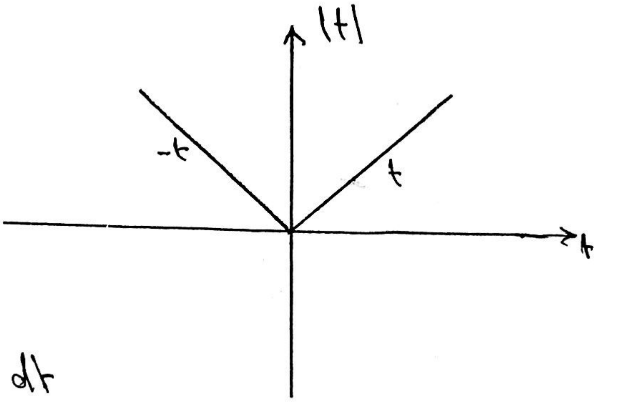
$$\mathcal{F}\{u(t)\} = \mathcal{F}\{\tfrac{1}{2}\} + \mathcal{F}\{\tfrac{1}{2}\,\mathrm{sgn}(t)\}$$

$$= \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathrm{sgn}(t)\}$$

$$= \frac{1}{2} \times 2\pi\,\delta(\omega) + \frac{1}{2} \times \frac{2}{j\omega}$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}$$

7- Absolute time $|t|$



$$|t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases}$$

$$\mathcal{F}\{|t|\} = \int_{-\infty}^{\infty} |t|\, e^{-j\omega t}\, dt$$

$$= -\int_{-\infty}^{0} t\, e^{-j\omega t}\, dt + \int_{0}^{\infty} t\, e^{-j\omega t}\, dt$$

let $u = t \Rightarrow du = dt$ & $dv = e^{-j\omega t}\,dt \Rightarrow v = \dfrac{e^{-j\omega t}}{-j\omega}$

$$\therefore \mathcal{F}\{|t|\} = -\left[\frac{t\,e^{-j\omega t}}{-j\omega}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{-j\omega t}}{-j\omega}\,dt\right] + \frac{t\,e^{-j\omega t}}{-j\omega}\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-j\omega t}}{-j\omega}\,dt$$

$$= -\left[[0-0] - \left[\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{-\infty}^{0}\right] + [0-0] - \left[\frac{e^{-j\omega t}}{(j\omega)^2}\right]_{0}^{\infty}$$

$$= \frac{1-0}{(j\omega)^2} + \frac{-(0-1)}{(j\omega)^2}$$

$$= \frac{2}{(j\omega)^2} = \frac{-2}{\omega^2}$$

Table: Some Fourier Transform Pairs.

Name	$f(t)$	$F(jw)$
Impulse	$\delta(t)$	1
Constant	A	$2\pi A\delta(w)$
Signum	$sgn(t)$	$\frac{2}{jw}$
step	$u(t)$	$\pi\delta(w)+\frac{1}{jw}$
post_t exponential	$e^{-at}u(t)$, $a>0$	$\frac{1}{a+jw}$
Neg_t exponential	$e^{at}u(-t)$, $a>0$	$\frac{1}{a-jw}$
Abs_t exponential	$e^{-a\|t\|}$, $a>0$	$\frac{2a}{a^2+w^2}$
Complex exponential	$e^{\mp jw_0 t}$	$2\pi\delta(w\pm w_0)$
Cosine	$\cos w_0 t$	$\pi[\delta(w+w_0)+\delta(w-w_0)]$
sine	$\sin w_0 t$	$+j\pi[\delta(w+w_0)-\delta(w-w_0)]$
Absolute time	$\|t\|$	$\frac{-2}{w^2}$
Squire Exponential	e^{-at^2}	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
product of time and Exp.	$t^n \cdot e^{-at}u(t)$	$\frac{n!}{(a+jw)^{n+1}}$

Properties of F.T

1. Linearity

$$\mathcal{F}\{a_1 f_1(t) \pm a_2 f_2(t)\} = a_1 F_1(\omega) \pm a_2 F_2(\omega)$$

2. Complex conjugate

$$\mathcal{F}\{f^*(t)\} = F^*(-\omega)$$

Proof:

$$\mathcal{F}\{f^*(t)\} = \int_{-\infty}^{\infty} f^*(t)\, e^{-j\omega t}\, dt = \left[\int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt\right]^* = F^*(-\omega)$$

IF $f(t)$ is real-valued, then $f^*(t) = f(t)$ and $F^*(\omega) = F(\omega)$

3. Scaling property :-

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Proof:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

let $z = at$, $dz = a\, dt$

$$= \int_{-\infty}^{\infty} f(z)\, e^{-j\omega z/a}\, \frac{dz}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(z)\, e^{-j\left(\frac{\omega}{a}\right) z}\, dz$$

$$= \frac{1}{a} F\left(\frac{\omega}{a}\right) \quad \text{for } a > 0$$

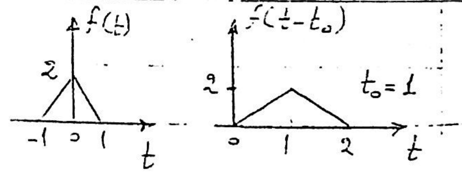
$$= \frac{-1}{a} F\left(\frac{\omega}{a}\right) \quad \text{for } a < 0$$

$$\therefore \mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Ex find $\mathcal{F}\{f(2t)\} = \frac{1}{|2|} F\left(\frac{\omega}{2}\right)$

4. Time Shifting (Delay)

$$\mathcal{F}\{f(t \mp t_o)\} = e^{\mp j\omega t_o} F(\omega)$$



Proof

$$\mathcal{F}\{f(t-t_o)\} = \int_{-\infty}^{\infty} f(t-t_o)\, e^{-j\omega t}\, dt$$

$z = t - t_o, \quad dz = dt$

$$= \int_{-\infty}^{\infty} f(z)\, e^{-j\omega(z+t_o)}\, dz = e^{-j\omega t_o} \int_{-\infty}^{\infty} f(z)\, e^{-j\omega z}\, dz = e^{-j\omega t_o} F(\omega)$$

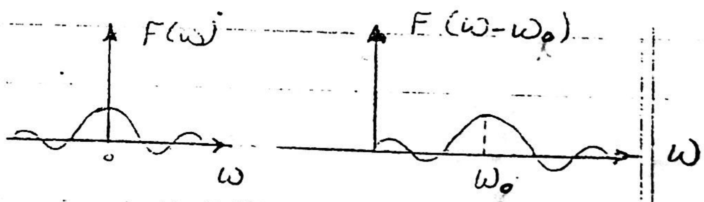
Ex $\mathcal{F}\{\delta(t-t_o)\} = e^{-j\omega t_o} \cdot 1$

$\boxed{5.}$ Frequency shifting (modulation)

$$\mathcal{F}\{f(t)\, e^{\pm j\omega_o t}\} = F(\omega \mp \omega_o)$$

Proof :

$$\because f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$



$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega-\omega_o)\, e^{j\omega t}\, d\omega$$

$z = \omega - \omega_o, \quad dz = d\omega$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(z)\, e^{j(z+\omega_o)t}\, dz = e^{j\omega_o t} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(z)\, e^{jzt}\, dz = e^{j\omega_o t} f(t)$$

Ex Find $\mathcal{F}\{f(t) \cos \omega_o t\}$

$$f(t) \cos\omega_o t = \frac{1}{2} f(t) \{e^{j\omega_o t} + e^{-j\omega_o t}\}$$

$$\therefore \mathcal{F}\{f(t) \cos\omega_o t\} = \frac{1}{2} \{F(\omega-\omega_o) + F(\omega+\omega_o)\}$$ modulation Property

Find $\mathcal{F}\left\{ \left(\frac{-t}{\sqrt{2}}\right)^2 \cdot f(t) \right\}$

$s = j\omega$

$\boxed{6}$ Differentiation Property

$$\mathcal{F}\left\{ \frac{d}{dt} f(t) \right\} = j\omega\, F(\omega)$$

$$\mathcal{F}\left\{ \frac{d^n}{dt^n} f(t) \right\} = (j\omega)^n\, F(\omega)$$

Proof; To prove

$$\mathcal{F}\{ f'(t) \} = j\omega\, F(\omega)$$

$$\because f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

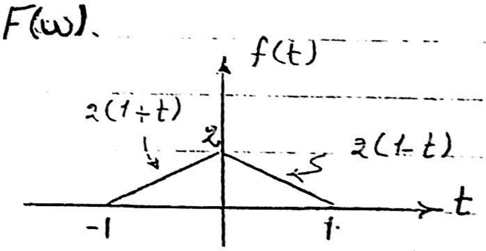
$\rightarrow$ as $f(t)$

$$f'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, j\omega\, d\omega$$

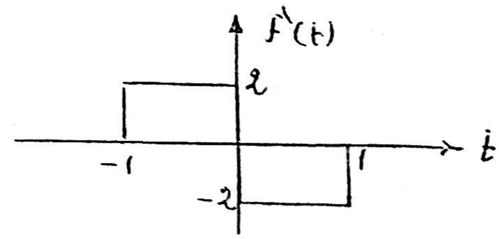
Taking F.T of both sides;

$$\mathcal{F}\{ f'(t) \} = j\omega\, \mathcal{F}\{ f(t) \} = j\omega\, F(\omega).$$

Ex Use the differentiation property to find $F(\omega)$ for the fun shown.

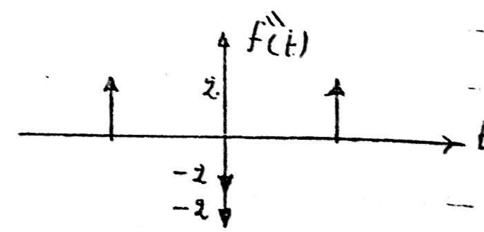


$$f''(t) = 2\,\delta(t+1) + 2\,\delta(t-1) - 4\,\delta(t)$$



$$(j\omega)^2 F(\omega) = 2e^{j\omega} + 2e^{-j\omega} - 4$$

$$F(\omega) = \frac{2(e^{j\omega} + e^{-j\omega}) - 4}{(j\omega)^2}$$

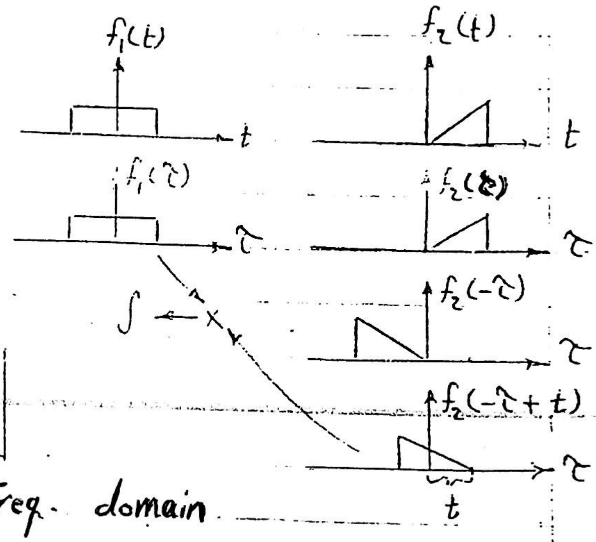


$$F(\omega) = \frac{4}{\omega^2}(1 - \cos\omega).$$

7.) Integration Property

$$\mathcal{F}\left\{ \int_{-\infty}^{t} f(t)\,dt \right\} = \frac{F(\omega)}{j\omega}$$

8.) Convolution in time



$$f_1(t) \circledast f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau)\,d\tau$$

$$\boxed{\mathcal{F}\{f_1(t) \circledast f_2(t)\} = F_1(\omega) \cdot F_2(\omega)}$$

$\therefore$ Conv. in time = multiplication in Freq. domain

Proof:

$$\mathcal{F}\{f_1(t) \circledast f_2(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,d\tau \right] e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \underbrace{\left\{ \int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-j\omega t}\,dt \right\}}_{\text{shift in time (4)}} d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\, F_2(\omega)\, e^{-j\omega\tau}\,d\tau$$

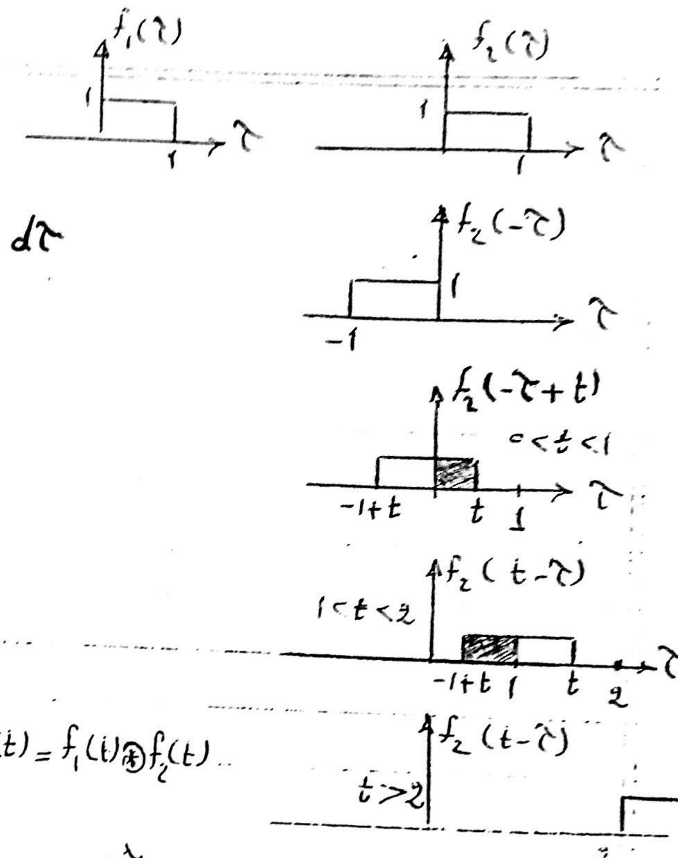
$$= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau)\, e^{-j\omega\tau}\,d\tau$$

$$= F_2(\omega) \cdot F_1(\omega)$$

Notes

1. $f_1(t) \circledast f_2(t) = f_2(t) \circledast f_1(t)$
2. $f_1(t) \circledast [f_2(t) + f_3(t)] = f_1(t) \circledast f_2(t) + f_1(t) \circledast f_3(t)$
3. $f_1(t) \circledast [f_2(t) \circledast f_3(t)] = [f_1(t) \circledast f_2(t)] \circledast f_3(t)$

Ex Find $f_1(t) \circledast f_2(t)$



تم تسمية المحاور مباشرة .

$$\underbrace{f_1(t) \circledast f_2(t)}_{\text{as } f(t)} = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

1. $0 < t < 1$

$$f(t) = \int_0^t 1.1\, d\tau = t$$

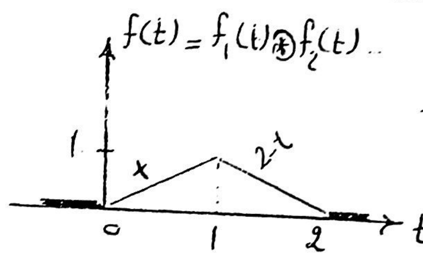
2. $1 < t < 2$

$$f(t) = \int_{t-1}^{1} 1.1\, d\tau = 2 - t$$

3. $t > 2$

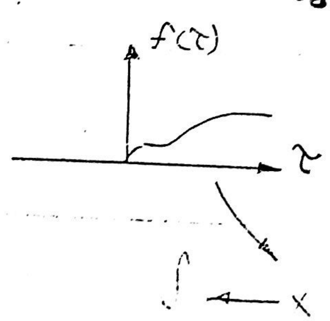
$$f(t) = 0$$

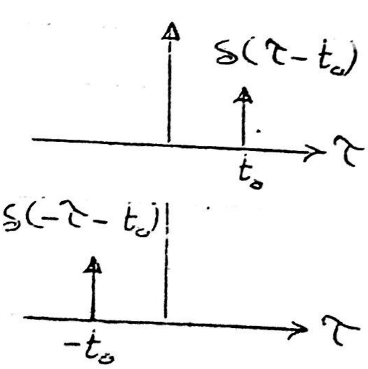
$$\therefore f(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 2-t & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$



Ex show $f(t) \circledast \delta(t-t_0) = f(t-t_0)$

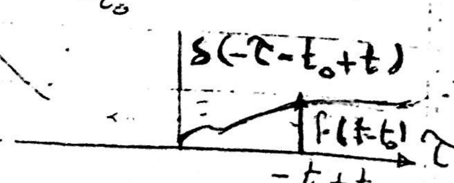
تم تسمية المحاور مباشرة .





$$\therefore f(t) \circledast \delta(t-t_0) = \int_{-\infty}^{\infty} f(\tau)\, \delta(t-t_0-\tau)\, d\tau$$

$$= f(t-t_0)$$



Parsevals theorem

$$W = \int_{-\infty}^{\infty} P(t)\, dt$$

$P(t) \longrightarrow$ power

$W \longrightarrow$ energy :

where $P(t) = V^2(t) = i^2(t) = f^2(t)$ at $R = 1\,\Omega$

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt$$

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(w)|^2\, dw \leftarrow \text{parsevals theorem}$$

Proof

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dw\right] dt$$

$$W_{1\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, F(w)\, e^{jwt}\, dw\, dt$$

Reversing the order of integration:-

$$W_{1\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w) \left[\int_{-\infty}^{\infty} f(t)\, e^{-j(-w)t}\, dt\right] dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, F(-w)\, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, F(w)^{*}\, dw$$

$$\Rightarrow \boxed{W_{1\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(w)^2|\, dw}$$

Ex The voltage across a $10\,\Omega$ resistor is $v(t) = 5e^{-3t}u(t)$. Find the total energy dissipated in the resistance.

$f(t) = v(t)$ or $F(w) = V(w)$

In the frequency domain

$$F(w) = V(w) = \frac{5}{3+jw}$$

So that $|F(w)|^2 = F(w) * F(w)^* = \frac{25}{9+w^2}$

that energy dissipated is :-

$$W_{10\Omega} = \frac{10}{2\pi}\int_{-\infty}^{\infty} |F(w)|^2\, dw = \frac{10}{\pi}\int_{0}^{\infty} \frac{25}{9+w^2}\, dw$$

$$= \frac{-250}{\pi}\left(\frac{1}{3}\tan^{-1}\frac{w}{3}\right)_0^{\infty} = \frac{250}{\pi}\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)$$

$$= \frac{250}{6} = 41.67\ \text{J}.$$

Ex: Find the total energy dissipated in a $10\text{-}\Omega$ resistance by the current $i(t) = 5e^{-10^3 t}\,u(t)$ A

$$W_{10\Omega} = 10 W_{1\Omega}$$

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_0^{\infty} 25\, e^{-2*10^3 t}\,dt$$

$$= \left[25 \frac{e^{-2*10^3 t}}{-2*10^3} \right]_0^{\infty} = 12.5 \text{ mJ}$$

$$\therefore W_{10\Omega} = 10\, W_{1\Omega} = 10 * 12.5 \text{ mJ} = 125 \text{ mJ} = 0.125 \text{ J}$$

H.W.8. Use Parseval's theorem to find $W_{1\Omega}$ if $f(t) = A e^{-a|t|}$

Ans: $\frac{A^2}{a}$

Ex: Evaluate $\int_{-\infty}^{\infty} \frac{4a^2}{(a^2+w^2)^2}$ for $a \geq 0$

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(jw)|^2\,dw$$

$$\int_{-\infty}^{\infty} \left(e^{-a|t|}\right)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2a}{A(a^2+w^2)}\right)^2 dw$$

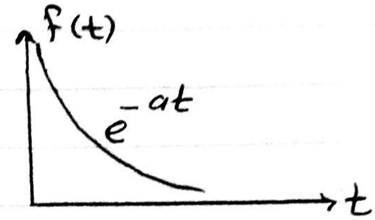
$$\int_{-\infty}^{0} e^{2at} + \int_0^{\infty} e^{-2at}\,dt = \frac{1}{2\pi}\int \frac{4a^2}{(a^2+w^2)^2}\,dw$$

$$\left[\frac{e^{2at}}{2a}\right]_{-\infty}^{0} + \left[\frac{e^{-2at}}{-2a}\right]_0^{\infty} = \frac{1}{2\pi}\int \frac{4a^2}{(a^2+w^2)^2}\,dw$$

$$\frac{1}{2a} + \frac{1}{2a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4a^2}{(a^2+w^2)^2} \Rightarrow \int_{-\infty}^{\infty} \frac{4a^2}{(a^2+w^2)^2} = \frac{2\pi}{a}$$

Ex: Obtain the F.T. of the switch-on exponential function below.

$$f(t) = e^{-at} = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}$$



$$F(jw) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

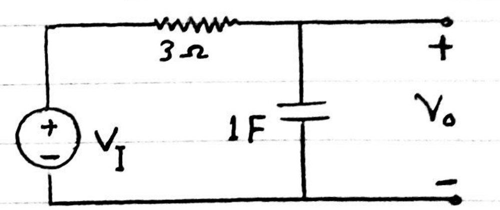
$$= \int_{-\infty}^{\infty} e^{-at} e^{-jwt}\, dt = \int_{0}^{\infty} e^{-(a+jw)t}\, dt$$

$$= \frac{-1}{a+jw} e^{-(a+jw)t} \Big|_{0}^{\infty} = \frac{-1}{a+jw} [\underbrace{\cancel{e^{-\infty}}}_{zero} - e^{0}] = \frac{1}{a+jw}$$

Ex: Using the F.T. approach, find the response $V_o(t)$ of the circuit of fig. to the applied signal $V_i(t) = 10 \cos 2t$ Volt

VDR:

$$V_o(jw) = \frac{1/jw}{3 + \frac{1}{1jw}} V_i(jw)$$



$$V_o(jw) = \frac{1}{3jw+1} V_i(jw) = \frac{1}{1+3jw} 10\pi [\delta(w+2) + \delta(w-2)]$$

because:-

$$V_i(t) = 10 \cos 2t \text{ V} \Rightarrow V_i(jw) = 10\pi [\delta(w+2) + \delta(w-2)]$$

$$V_o(t) = \mathcal{L}^{-1} V_o(jw) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 10\pi \frac{[\delta(w+2) + \delta(w-2)]}{1+3jw} e^{jwt}\, dw$$

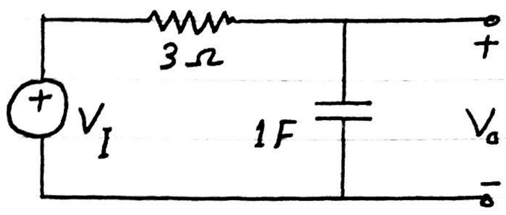
$$= 5\left(\frac{e^{j2t}}{1+6j} + \frac{e^{-j2t}}{1-6j}\right)$$

$$= \frac{10}{\sqrt{37}} \cos(2t - 80.54^\circ) \text{ Volt}$$

Ex: Repeat last example, if the circuit include also a 0.5 H inductance in parallel with the capacitance

Ans. $V_o(t) = \sqrt{10}\cos(2t - 71.57^\circ)$ volt $\boxed{\text{H.W. 9}}$

Ex: For the circuit:-



(a) If $V_I(t) = 10\, e^{-t/4}\, u(t)$ V
find w_i & w_o & η

Ans. $w_i = 200$ J & $w_o = 114.29$ J
$\eta = 57.14\,\%$

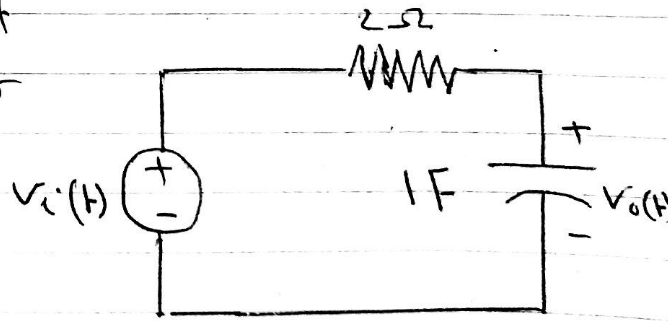
note: $\eta = \frac{w_o}{w_i} * 100\%$

(b) find w_i & w_o & η, if the resistance and capacitance interchanged with each other.

$\boxed{\text{H.W. 10}}$

Ans. $\eta = 42.86\,\%$

Ex: Find $v_o(t)$ in the circuit
For $v_i(t) = 2e^{-3t}u(t)$ by using Fourier Transform



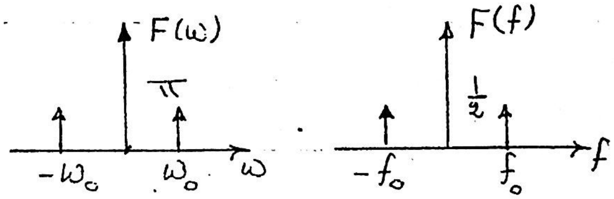
Application of F.T to Communication

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

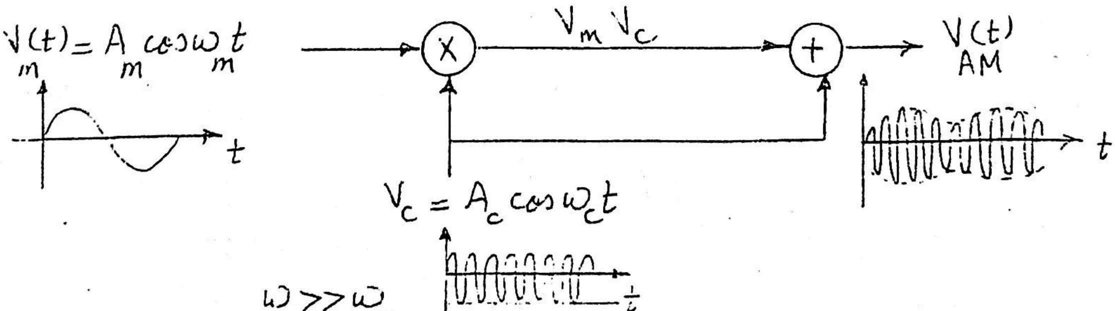
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \mathcal{F}\{f(t)\}$$

F.T

$f(t)$	$\mathcal{F}\{f(t)\}$, $F(\omega)$ $\quad \omega = 2\pi f$
$\delta(t)$	1
1	$2\pi\,\delta(\omega) = \delta(f)$
$\delta(t-t_o)$	$e^{-j\omega t_o}$
$e^{\pm j\omega_x t}$	$2\pi\,\delta(\omega \mp \omega_x) = \delta(f \mp f_x)$
$u(t)$	$\pi\,\delta(\omega) + 1/j\omega$
$\cos\omega_o t = \frac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})$	$\frac{1}{2}[\delta(f-f_o) + \delta(f+f_o)]$ or $\pi[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$



(AM) modulation



$\omega_c >> \omega_m$

$$V_{AM}(t) = V_c + V_m V_c$$

$$= A_c \cos\omega_c t + A_c A_m \cos\omega_m t \cos\omega_c t$$

$$= A_c \cos\omega_c t + \frac{A_c A_m}{2} \{ \cos(\overbrace{\omega_c - \omega_m}^{\omega_x}) t + \cos(\overbrace{\omega_c + \omega_m}^{\omega_z}) t \}$$

$$= \frac{A_c}{2} \{ e^{j\omega_c t} + e^{-j\omega_c t} \} + \frac{A_c A_m}{4} \{ e^{j\omega_x t} + e^{-j\omega_x t} + e^{j\omega_z t} + e^{-j\omega_z t} \}$$

$$\therefore \mathcal{F}\{V_{AM}(t)\} = \frac{A_c}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] + \frac{A_c A_m}{4}\left[\delta(f-f_x) + \delta(f+f_x)\right]$$

$$+ \frac{A_c A_m}{4}\left[\delta(f-f_z) + \delta(f+f_z)\right]$$

where $f_x = f_c - f_m$ and $f_z = f_c + f_m$